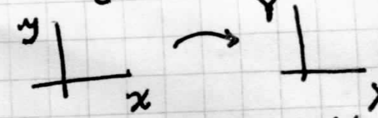
Hydon ch 2
p. 15–16

ch 2.1 The action of Lie Sym on $\mathbb{R}^2$ ①
(Htpy of soln curves to soln curves)

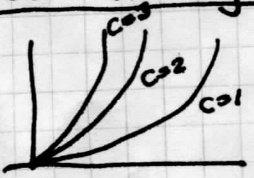
$S_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$



(ex 2.1 and 2.2) $\dfrac{dy}{dx} = \dfrac{2y}{x}$ The soln is $y(x) = cx^2$ as we can readily verify: $y' = 2cx = 2cx\frac{x}{x} = 2c\frac{x^2}{x} = 2y/x$

Consider only Quad I

we want $S : (\text{soln curves}) \to (\text{soln curves})$



$$S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto \begin{bmatrix} x/y \\ 1/y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

Here is a <u>discrete sym</u> (no ε)

Then a soln $y = cx^2$ gets mapped

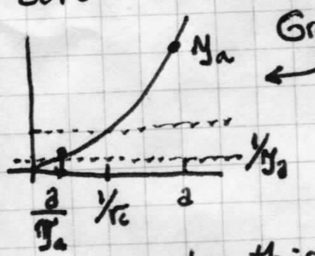
$$S : (x, cx^2) \longmapsto \begin{bmatrix} \frac{x}{cx^2} \\ \frac{1}{cx^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{cx} \\ \frac{1}{cx^2} \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix}$$

We <u>identify</u> the (x,y) and (X,Y) planes and we want to see if the soln curves of ODE coincide. That means does $Y = cX^2$?

$$X = \frac{1}{cx} \Rightarrow x = \frac{1}{cX} \quad \text{then } Y = \frac{1}{cx^2} = \frac{1}{c\left(\frac{1}{cX}\right)^2} = cX^2 \ \checkmark$$

$$\langle x, cx^2 \rangle \longmapsto \langle X, cX^2 \rangle$$

Let's look at this differently to be clearer:



Let $g_c(x) := cx^2$ Fix a pt $x = a$ (a.k.a x_a)

$g_c(a) = ca^2 = y_a$

So the pt on $Gr(g_c) = \langle a, g_c(a) \rangle = \langle a, ca^2 \rangle = \langle x_a, y_a \rangle$

apply $S : \langle x_a, y_a \rangle \longmapsto \langle \frac{x_a}{y_a}, \frac{1}{y_a} \rangle$

Is this new pt still on $Gr(g_c)$? Yes, because

$$g_c\left(\frac{x_a}{y_a}\right) = g_c\left(\frac{a}{ca^2}\right) = c\left(\frac{a}{ca^2}\right)^2 = \frac{1}{ca^2} = \frac{1}{y_a}$$

Thus $S : Gr(g_c) \longrightarrow Gr(g_c)$ invariant set under S

But S is not any sort of <u>flow</u> along this curve, rather some inversion.

Does S have a fixed pt? $S(a, g_c(a)) \stackrel{!}{=} \langle a, g_c(a) \rangle$

$$\begin{bmatrix} a/ca^2 \\ 1/ca^2 \end{bmatrix} = \begin{bmatrix} a \\ ca^2 \end{bmatrix} \Rightarrow \cancel{a} \stackrel{!}{=} \frac{\cancel{a}}{ca^2} \Rightarrow 1 = \frac{1}{ca^2} \Rightarrow a = \frac{1}{\sqrt{c}}$$

$$g_c(1/\sqrt{c}) = c\frac{1}{c} = 1 \qquad \langle 1/\sqrt{c}, 1 \rangle \xrightarrow{S} \begin{bmatrix} \frac{1/\sqrt{c}}{1} \\ 1/1 \end{bmatrix} = \langle 1/\sqrt{c}, 1 \rangle \quad \text{QED}$$

example cont'd ⟶

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ex 2.2 is a continuation

<u>Now consider this Sym, a <u>Lie Sym</u> with homotopy param</u>

$$S_\varepsilon(x,y) = \begin{bmatrix} e^{\varepsilon} & \\ & e^{-\varepsilon}\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = \begin{bmatrix} X \\ Y\end{bmatrix} \qquad X = e^{\varepsilon}x \Rightarrow x = e^{-\varepsilon}X$$

$$S_\varepsilon(x, cx^2) = \begin{bmatrix} e^{\varepsilon}x \\ e^{-\varepsilon}cx^2\end{bmatrix} = \begin{bmatrix} X \\ e^{-\varepsilon}c(e^{-\varepsilon}X)^2\end{bmatrix} = \begin{bmatrix} X \\ ce^{-3\varepsilon}X^2\end{bmatrix} = \begin{bmatrix} X \\ c_1X^2\end{bmatrix}$$

[If $\gamma(x) = \begin{bmatrix} x \\ y(x)\end{bmatrix}$ is a pt of a soln curve $\forall\, x \in I$

Then we want $S_\varepsilon(\gamma(x))$ to also be on a soln curve]

So this is another distinct soln curve, $y' = 2\frac{y}{x}$ This is a homog eq. The sym given here does <u>NOT</u> lead to soln

(p.18) I am going to call $S_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ the '<u>Sym flow</u>' and

If we fix x, we get a flow line thru x; the curve $\sigma_x(\varepsilon)$

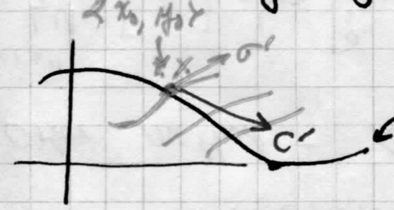
$$S_\varepsilon(x,y) = \sigma_x(\varepsilon) = \begin{bmatrix} X(\varepsilon) \\ Y(\varepsilon)\end{bmatrix} \qquad \text{Thus} \quad \sigma_x'(\varepsilon) = \begin{bmatrix} X'(\varepsilon) \\ Y'(\varepsilon)\end{bmatrix} =: \begin{bmatrix} \xi(X,Y) \\ \eta(X,Y)\end{bmatrix}$$

$| S^{\varepsilon=0} = Id$

Thus

$$X(0) = x \qquad X'(0) = \xi(x,y)$$
$$Y(0) = y \qquad Y'(0) = \eta(x,y)$$

If a pt $\langle x,y\rangle$ is not moved by sym flow S^ε (i.e. it is a FP) then we say $\langle x,y\rangle$ is <u>invariant</u> and $\xi(x,y)=0$, $\eta(x,y)=0$ ie $\sigma_x'(0) = \vec{0}$



Some curve $C(x) = \begin{bmatrix} x \\ y(x)\end{bmatrix}$

If the sym flow curve σ crosses curve C <u>transversally</u> at X then C is not held fixed by Sym flow.

BUT if C' is parallel to σ', then not $\pitchfork$

That means $\begin{bmatrix} \xi \\ \eta\end{bmatrix}$ is colinear with $c' = \begin{bmatrix} 1 \\ y'\end{bmatrix}$ ie not LI vectors

$$\text{ie } \det\begin{bmatrix} | & | \\ c' & \sigma' \\ | & | \end{bmatrix} = \det\begin{bmatrix} 1 & \xi \\ y' & \eta\end{bmatrix} = 0 \Rightarrow \underbrace{\eta - \xi y'}_{Q(x,y,y')} = 0 \quad \text{"characteristic"}$$

But if C not arb but rather the flow of ODE [and then $y' = w(x,y)$]

$$\bar{Q}(x,y) := \eta(x,y) - \xi(x,y)\,w(x,y)$$

Thus $\bar{Q} = 0 \Rightarrow \eta = \xi w$ (2.18)

key

[A soln curve $y = f(x)$ is held fixed by Sym flow] $\Longleftrightarrow$ $\bar{Q}(x,y) = 0$ $\forall x$ in interval.

Hydon p.20 9/8/2023

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Recall Imp Fcn thm: $F(x_0, y_0) = 0$ and $D_2 F \neq 0$ $\Rightarrow$ $\exists$ fcn $y = f(x)$ in nbhd of $\langle x_0, y_0 \rangle$ and $\frac{dF}{dx} = D_1 F + D_2 F y' = 0$ $\Rightarrow y' = -\frac{D_1 F}{D_2 F}$

[Here apply that to $\bar{Q}(x_0, y_0) = 0$ and $D_2 Q_{x_0} \neq 0$
Then we see a soln to the ODE exists locally]

The ODE admits a Lie sym S_ε, so it has this as a Sym flow
<u>If</u> this flow is tangent to ODE soln curves, then <u>sym flow is flowing along soln curves</u> $\Rightarrow$ σ_x is a soln curve to ODE!
and thus we have solved ODE.

ex 2.4 $y' = y$ (same as p.8)

They Say: This has Scaling Sym $\underset{\sigma_x(\varepsilon)}{S^\varepsilon(x,y)} = \begin{bmatrix} 1 & \\ & e^\varepsilon \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ e^\varepsilon y \end{bmatrix}$ Thus $X = x = S^{(1)}(x,y)$, $Y = e^\varepsilon y = S^{(2)}(x,y)$

[ASIDE:
How do we know this is really a Lie Sym? Need to see $\frac{dY}{dX} = Y$ because $\frac{dy}{dx} = y$

$$\frac{dY}{dX} = \frac{dY/dx}{dX/dx} = \frac{Y_x + Y_y y'}{X_x + X_y y'} \overset{!}{=} W(X,Y) = Y \Rightarrow \frac{0 + e^\varepsilon y}{1 + 0} \overset{!}{=} e^\varepsilon y \ \checkmark$$
]

So we had $\sigma_x(\varepsilon) = \begin{bmatrix} x \\ e^\varepsilon y \end{bmatrix}$ then $\sigma_x'(\varepsilon) = \begin{bmatrix} 0 \\ e^\varepsilon y \end{bmatrix}$ and $\sigma_x'(0) = \begin{bmatrix} 0 \\ y \end{bmatrix} \overset{!}{=} \begin{bmatrix} \xi(x) \\ \eta(x) \end{bmatrix}$ $x = \langle x_0, y_0 \rangle$

Recall (2.13) x invariant (held fixed) under S^ε $\Leftrightarrow$ $\xi(x) = 0$, $\eta(x) = 0$

so for a FP here, we must have $\eta = 0$ i.e. $y = 0$

$$\bar{Q}(x,y) = \eta(x,y) - \xi(x,y)\, W(x,y) = y - 0 \cdot y = y$$

Thus $\bar{Q} \neq 0$ (unless we are on x axis)
$\Rightarrow S^\varepsilon$ does <u>not</u> hold fixed soln curves,
(except for the one $y(x) = 0$)
we know the soln are $y = ce^x$

Here is another sym they give (again not from p.8-a)

$$S_\varepsilon(x,y) = \begin{bmatrix} e^\varepsilon & \cdot \\ \cdot & e^{(e^\varepsilon - 1)x} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad X = e^\varepsilon x, \quad Y = e^{(e^\varepsilon - 1)x}$$

NOTE: $\frac{d}{d\varepsilon} e^{f(\varepsilon)} = e^{f(\varepsilon)} f'(\varepsilon)$

$Y'(\varepsilon) = e^{(\varepsilon - 1)x} e^\varepsilon x$

$$\sigma'(\varepsilon) = \begin{bmatrix} X'(\varepsilon) \\ Y'(\varepsilon) \end{bmatrix} = \begin{bmatrix} e^\varepsilon x \\ \underbrace{e^{(e^\varepsilon - 1)x}}_{Y} \underbrace{e^\varepsilon x}_{X} \end{bmatrix} = \begin{bmatrix} X \\ YX \end{bmatrix}$$

?

then $\sigma'(0) = \begin{bmatrix} x \\ x \end{bmatrix} \neq [$

ex 2.5 a particular Ricatti eq: $y' = \frac{-1}{x^3} - \frac{2}{x}y + xy^2 \quad x \neq 0$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = S_\varepsilon(x,y) = \begin{bmatrix} e^\varepsilon & \\ & e^{-2\varepsilon} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^{\varepsilon}x \\ e^{-2\varepsilon}y \end{bmatrix}$$

For fixed $x = \langle x, y \rangle$ The sym flow line is $\sigma_x(\varepsilon) = S_\varepsilon(x,y)$

Then $\sigma_x'(\varepsilon) = \begin{bmatrix} e^\varepsilon x \\ -2e^{-2\varepsilon}y \end{bmatrix} \quad \sigma_x'(0) = \begin{bmatrix} x \\ -2y \end{bmatrix} =: \begin{bmatrix} \xi(x) \\ \eta(x) \end{bmatrix}$

To have sym flow along soln curves of ODE we must have $\bar{Q} = 0$

$$\bar{Q}(x,y) = \eta(x) - w(x,y)\,\xi(x)$$
$$= -2y - \left(xy^2 - \frac{2}{x}y - \frac{1}{x^3}\right)(x) = \cancel{-2y} - x^2y^2 + \cancel{2y} + \frac{1}{x^2}$$
$$= -x^2y^2 + \frac{1}{x^2} \quad \text{and we want } \bar{Q} = 0$$

$$\Rightarrow \frac{1}{x^2} - x^2y^2 \overset{!}{=} 0 \Rightarrow x^2 y^2 = \frac{1}{x^2}$$
$$y^2 = \frac{1}{x^4} \Rightarrow y(x) = \pm\frac{1}{x^2}$$

Thus there are 2 invariant solns $y_{\oplus}(x) = \frac{1}{x^2}$ and $y_{\ominus}(x) = -\frac{1}{x^2}$

ASIDE:
Is $y = \frac{1}{x^2} = x^{-2}$ really a soln?
$$y' \overset{?}{=} -x^{-3} - \frac{2}{x}y + xy^2$$
$$-2x^{-3} = \cancel{-x^{-3}} - 2x^{-3} + \cancel{x^{-3}} \checkmark$$

see 10/10/23 sheet for Hydon p. 27

ex 2.6 Given $\sigma_x'(\varepsilon)$, can we recover $\sigma_x(\varepsilon) = S_\varepsilon(x)$? maybe integrate!

Specifically for ex 2.4 $\quad \sigma_x'(\varepsilon) = \begin{bmatrix} 0 \\ e^\varepsilon y \end{bmatrix} = \begin{bmatrix} X'(\varepsilon) \\ Y'(\varepsilon) \end{bmatrix}$ and at the initial pt $\varepsilon=0$

By inspecting, X does not depend on ε so $X = X(x,y)$ $\quad X(x,y) = x$

$Y'(\varepsilon) = e^\varepsilon y \rightarrow \int Y'(\varepsilon)\,d\varepsilon = y\int e^\varepsilon d\varepsilon + \varphi(x,y)$

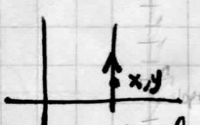
$Y(x,y,\varepsilon) = ye^\varepsilon + \varphi(x,y)$ But $Y(x,y,0) = y$

so $\varphi(x,y) \equiv 0$.

In ch 1.4 we showed that if the ODE has a translation sym $\begin{bmatrix} X \\ Y \end{bmatrix} = S_\varepsilon(x,y) = \begin{bmatrix} x \\ y+\varepsilon \end{bmatrix}$ $\sigma_x(\varepsilon)$

THEN we can reduce to quadrature.

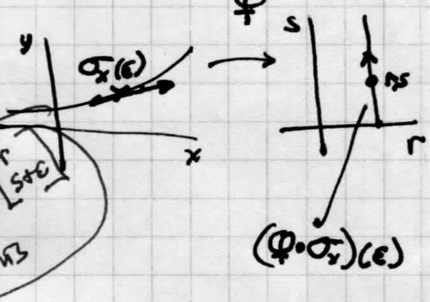
MORE GENERALLY — can we find a COV to make this happen?



vertical sym flow

$$\sigma_x(\varepsilon) = \begin{bmatrix} x \\ y+\varepsilon \end{bmatrix} \Rightarrow \sigma_x'(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

we aim to introduce co-ords $\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} r(x,y) \\ s(x,y) \end{bmatrix}$



such that $\begin{bmatrix} R \\ S \end{bmatrix} = \begin{bmatrix} r(X,Y) \\ s(X,Y) \end{bmatrix} = \begin{bmatrix} r \\ s+\varepsilon \end{bmatrix}$

Bad to be using same letter.

$$= \begin{bmatrix} r(S_\varepsilon(x,y)) \\ s(S_\varepsilon(x,y)) \end{bmatrix}$$

$\varphi \circ \sigma_x(\varepsilon) \overset{!}{=} \begin{bmatrix} r \\ s+\varepsilon \end{bmatrix}$ want this

If we can do this, then tangent vector at (r,s) is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

KEY $\dfrac{dR}{d\varepsilon}(\varepsilon=0) = 0 \qquad \dfrac{dS}{d\varepsilon}(0) = 1$ that is $\dfrac{d}{d\varepsilon}\Big|_{\varepsilon=0} (\varphi\circ\sigma) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

By calculus: $\dfrac{d}{d\varepsilon}(\varphi\circ\sigma)\Big|_{\varepsilon=0} = D\varphi_{\sigma(0)}\,\sigma'(0) = \begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix} \begin{bmatrix} \xi(x,y) \\ \eta(x,y) \end{bmatrix} \overset{!}{=} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (2.28)

and we require $\det(D\varphi_{\sigma(0)}) \neq 0$

so φ^{-1} exists locally

This means $r=$ const and $s=$ const intersect π

But the goal is to solve for r, S

?   By def the tan vector at any non-FP is parallel to curve $r=$ const

At a FP of sym flow $\xi = 0$, $\eta = 0$ so $\begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

p. 19-20 "Method of characteristics"

$\det\begin{bmatrix} 1 & \xi \\ y' & \eta \end{bmatrix} = 0$

i.e. $\eta - y'\xi = 0$

i.e. $\eta - \omega\xi = 0$ ($y' = \omega$ on soln curve)

p. 24 Solns $(r(x,y), s(x,y))$ to (2.28) are not unique:

$(\bar r, \bar s) := (F(r), s + G(r))$ would also work ($\det D\varphi \neq 0 \rightarrow F'(r) \neq 0$)

We wish to make $r(x,y)$, $s(x,y)$ as simple as possible:

Usually try $\begin{cases} \eta \text{ linear in } y \\ \xi \text{ indep of } y \end{cases}$

No, I think this is: we find ξ, and η in many cases

Hydon p.24
Starrett p.16

9/29/2023

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we seek to solve $\begin{cases} r_x \xi + r_y \eta = 0 & ⓐ \\ s_x \xi + s_y \eta = 1 & ⓑ \end{cases}$ we solve for fcns $\underline{r = r(x,y)}$, $\underline{s = s(x,y)}$ Bad Notation, use $\mathcal{S}$

(2.28) Hydon (7) Starret

▷ How can we solve ⓐ?

$r_x \xi + r_y \eta = 0$ assume $\xi \neq 0$

$r_x + r_y \dfrac{\eta}{\xi} = 0$

Now we observe for the ODE $y' = f(x,y)$ a $\underline{1^{st} \text{ Integral}}$ φ is const on soln curves $y = y(x)$

$\dfrac{d}{dx} \varphi(x, y(x)) = c \quad \dfrac{d}{dx} \Rightarrow \quad \varphi_x + \varphi_y y' = 0$ $\qquad \varphi(x, y(x)) = c$

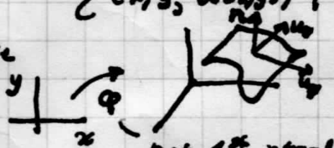
By comparison $y' = \dfrac{dy}{dx} = \dfrac{\eta}{\xi}$ or $\dfrac{dy}{\eta} = \dfrac{dx}{\xi}$ (2.35) "characteristic eq"

▷ How can we solve ⓑ? Here we need Starrett:

ⓑ $\Rightarrow s_x \xi + s_y \eta - 1 = 0$ This is a pde of the form $\boxed{a u_x + b u_y - c = 0}$ (★)

For a fcn $z = u(x,y)$ the graph $Gr(u)$ is surf $\mathcal{S}$ $= \{ (x, y, \overset{z}{u(x,y)}) \mid (x,y) \in D \}$

This surf $\mathcal{S}$ has normal $\vec{n} = \begin{bmatrix} -u_x \\ -u_y \\ 1 \end{bmatrix}$ because



not 1^{st} integral from above

$\vec{\varphi}_x \times \vec{\varphi}_y = \det \begin{vmatrix} i & j & k \\ 1 & 0 & u_x \\ 0 & 1 & u_y \end{vmatrix} = \begin{bmatrix} -u_x \\ -u_y \\ 1 \end{bmatrix}$

Then ★ can be expressed as

$[a \; b \; c] \begin{bmatrix} -u_x \\ -u_y \\ 1 \end{bmatrix} = 0$ (mult by -1)

Then $\vec{a}(x) \perp \vec{n}(x)$ so $\vec{a} \in T_x \mathcal{S}$

This means that at the pt x, $\vec{a}$ is the tangent vector to a curve $\gamma \in \mathcal{S}$ (maybe γ only in $\mathcal{S}$ at pt x)

If we write $\gamma(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$ then $\gamma'(t) = \begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow$

$\Rightarrow \dfrac{dx}{dt} = a \quad dy/dt = b \quad dz/dt = c \quad \Rightarrow dt = \dfrac{dx}{a} = \dfrac{dy}{b} = \dfrac{dz}{c}$

For our eq ⓑ $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \\ 1 \end{bmatrix} \Rightarrow \dfrac{dx}{\xi} = \dfrac{dy}{\eta} = \dfrac{dz}{1} = dz$ rename "z" as "$\mathcal{S}$"

Now Hydon p.24 has $d\mathcal{S} = \dfrac{1}{\xi(x,y)} dx \Rightarrow \mathcal{S}(r, x) = \displaystyle\int \frac{1}{\xi(x, y(r,x))} dx \Big|_{r = r(x,y)}$

Lets do an example to make this clearer!

9/29/2023

NO ODE specified

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Hydon p.25 example 2.7

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} e^{\varepsilon} & \\ & e^{k\varepsilon} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^{\varepsilon} x \\ e^{k\varepsilon} y \end{bmatrix} = \sigma_x(\varepsilon) \qquad \sigma_x'(\varepsilon) = \begin{bmatrix} e^{\varepsilon} x \\ k e^{k\varepsilon} y \end{bmatrix}$$

we seek fcns $r, s \ni$

$$\begin{bmatrix} r_x & r_y \\ s_x & s_y \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{matrix} ⓐ \\ ⓑ \end{matrix} \qquad \sigma_x'(0) = \begin{bmatrix} x \\ k y \end{bmatrix} = \begin{bmatrix} \xi(x,y) \\ \eta(x,y) \end{bmatrix}$$

eq ⓐ $r_x \xi + r_y \eta = 0 \Rightarrow \frac{dy}{\eta} = \frac{dx}{\xi}$ or $\frac{dy}{dx} = k \frac{y}{x}$ $\quad \int \frac{1}{y} dy = k \int \frac{1}{x} dx$

If $r(x,y) := x^{-k} y$ then for this soln $y(x) = C x^k$ $\quad \Rightarrow y(x) = C x^k$

$r(x, y(x)) = x^{-k}(c x^k) = c$ const on soln curves $\langle x, y(x) \rangle$ $\rightarrow$ 1st integral

Then we would also like to solve eq ⓑ $s_x \xi + s_y \eta = 1$

from prev page $\frac{dx}{\xi} = \frac{dy}{\eta} = ds$ so $ds = \frac{1}{\xi(x,y)} dx = \frac{1}{x} dx$

$\Rightarrow s = \ln|x|$ on $\mathbb{R} - \{0\}$

so we should have $\begin{bmatrix} r(S_\varepsilon(x,y)) \\ s(S_\varepsilon(x,y)) \end{bmatrix} = \begin{bmatrix} r \\ s + \varepsilon \end{bmatrix}$ Lets verify:

$$r(S_\varepsilon(x,y)) = r(X,Y) = X^{-k} Y = (e^{\varepsilon} x)^{-k} e^{k\varepsilon} y = \cancel{e^{-k\varepsilon}} x^{-k} \cancel{e^{k\varepsilon}} y$$
$$= x^{-k} y = r(x,y) \checkmark$$

$$s(S_\varepsilon(x,y)) = s(X,Y) = \ln|X|$$
$$= \ln|e^{\varepsilon} x| = \ln e^{\varepsilon}|x|$$
$$= \ln e^{\varepsilon} + \ln|x| = \varepsilon + s \checkmark$$

$\square$

skipping ex 2.8 for now

& 2.9

Hydon p. 27
Ruth p. 37 10/10/2023

ex 2.10 a Riccati eq from p.21 $\overset{\text{ODE}}{y'} = \frac{-1}{x^3} - \frac{2}{x}y + xy^2$ $x \neq 0$ (2.50)

has the Lie Syms $S_\varepsilon(x,y) = \begin{bmatrix} e^\varepsilon & \\ & e^{-2\varepsilon} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ This is the same Sym as ex 2.7 p.25 with $k=-2$

$$\sigma_x'(0) = \begin{bmatrix} x \\ -2y \end{bmatrix} =: \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

In ex 2.5 p.21 we found <u>Solns to the ODE</u> that are <u>invariant</u> (held fixed) by S_ε

<u>Now we shall seek a larger set of Solns – a 1 param family that is moved by S_ε</u>

First, recap ex 2.5 : To have the Sym flow <u>along</u> Soln curves of ODE, we must have $\bar{Q} = 0$:

$$\bar{Q}(x,y) = \eta(x,y) - \omega(x,y)\xi(x,y) \overset{!}{=} 0$$
$$= -2y - \left(xy^2 - \frac{2}{x}y - \frac{1}{x^3}\right)(x) = -x^2y^2 + \frac{1}{x^2} \overset{!}{=} 0 \Rightarrow y(x) = \frac{\pm 1}{x^2}$$

$y_\oplus = \frac{1}{x^2}$ $y_\ominus = \frac{-1}{x^2}$

From ex 2.7 with $k=-2$ we have $r(x,y) = x^{-k}y = x^{-(-2)}y = x^2 y$
$s(x,y) = \ln|x|$

we want to compute

$$(2.46) \quad \frac{ds}{dr} = \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y}$$

$r_x = 2xy$ $s_x = 1/x$
$r_y = x^2$ $s_y = 0$

Ruth Steinhour p.37

$$= \frac{1/x + (\cdots)0}{\cancel{2xy} + \underbrace{\left(xy^2 - \frac{2}{x}y - \frac{1}{x^3}\right)(x^2)}_{x^3y^2 - \cancel{2xy} - 1/x}}$$

$$= \frac{\frac{1}{x}}{x^3y^2 - \frac{1}{x}}\,\frac{x}{x} = \frac{1}{x^4y^2 - 1}$$

Then by
$r = x^2 y$
$s = \ln|x| \Rightarrow e^s = x$
$x = e^s$
$y = r e^{-2s}$

$$\frac{ds}{dr} = \frac{1}{r^2-1} \qquad = \frac{1}{r^2-1}$$

$$\int ds = \int \frac{1}{r^2-1}\,dr$$

By partial fractions (common denominators backwards) we obtain : [See next page for details →]

$$s = \frac{1}{2}\ln\left(\frac{r-1}{r+1}\right) + k_0$$

subs back for x, y:

$$y_k = \frac{-(x^2+k)}{(x^4 - kx^2)}$$ This is a 1 param family of solns of ODE

when $k=0$ $y_0(x) = \frac{-x^2}{x^4} = \frac{-1}{x^2} = y_\ominus$ invariant Soln found previously

when $k \to \infty$ $\lim_{k\to\infty} \frac{-(x^2+k)}{(x^4-kx^2)} \longrightarrow \frac{-1}{-x^2} = \frac{1}{x^2} = y_\oplus$ found previously

□

Ruth p.37 — Partial fracs — on 10/10/2023 Sheet (2)

$$s = \int \frac{1}{r^2-1}\,dr = \frac{1}{2}\int \frac{1}{r-1}\,dr - \frac{1}{2}\int \frac{1}{r+1}\,dr$$

$$\frac{1}{(r+1)(r-1)} = \frac{A}{(r+1)} + \frac{B}{(r-1)} = \frac{(r-1)A}{(r-1)(r+1)} + \frac{B(r+1)}{(r-1)(r+1)}$$

$$\Rightarrow (r-1)A + B(r+1) \overset{!}{=} 1$$

$$Ar - A + Br + B$$

$$\Rightarrow (A+B)r + (B-A)$$

so $A+B \overset{!}{=} 0$ $\quad B-A \overset{!}{=} 1$

This has soln $B = 1/2$ $A = -1/2$

$$\frac{-1/2}{(r+1)} + \frac{1/2}{(r-1)}$$

$$\to \frac{1}{2}\left[\ln(r-1) - \ln(r+1)\right] + k_0 = \frac{1}{2}\ln\left(\frac{r-1}{r+1}\right) + k_0$$

subs $r = x^2 y$ $\quad s = \ln x$

$$s = \frac{1}{2}\ln\left(\frac{x^2y-1}{x^2y+1}\right)$$

$$\ln x = \cancel{\tfrac{1}{2}} \ln\left(\frac{x^2y-1}{x^2y+1}\right)^{1/2} + k_0$$

raise e:

$$x = k\left(\frac{x^2y-1}{x^2y+1}\right)^{1/2} \qquad k^2 = k \text{ just rename it}$$

$$x^2 = k\left(\frac{x^2y-1}{x^2y+1}\right)$$

$$x^2(x^2y+1) = k(x^2y-1)$$

$$x^4y + x^2 = kx^2y - k$$

$$x^4y - kx^2y = -x^2 - k$$

$$\boxed{y_k = \frac{-(x^2+k)}{(x^4-kx^2)}}$$

This is the general soln 1 param family depending on k

when $k=0$

$$y(x) = \frac{-x^2}{x^4} = \frac{-1}{x^2}$$ soln we found previously $y_\ominus$

when $k \to \infty$

$$\lim_{k\to\infty} \frac{-(x^2+k)}{x^4-kx^2} = \frac{\infty}{\infty}$$ apply L'Hop $\frac{-1}{-x^2} = \frac{1}{x^2}$

$\Rightarrow$ soln we found previously $y_\oplus = \frac{1}{x^2}$

Hydon p. 30 ch 2.4 Linearized Symm Cond 10/3/2023

3

Sym cond (1.10) (2.54) $\dfrac{Y_x + \omega(x,y) Y_y}{X_x + \omega(x,y) X_y} \overset{!}{=} \omega(X,Y)$

(2.1) $\dfrac{dy}{dx} = \omega(x,y)$ $\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \varepsilon \begin{bmatrix} \xi(x,y) \\ \eta(x,y) \end{bmatrix} + O(\varepsilon^2)$

$S_\varepsilon(x,y) = \sigma_x(\varepsilon)$ $\sigma_x : \mathbb{R} \longrightarrow \mathbb{R}^2$, $\varepsilon \longmapsto$

(2.55) Taylor expand:

$\sigma_x(\varepsilon) = \sigma_x(0) + \sigma_x'(0)(\varepsilon - 0) + O(\varepsilon^2)$

$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \varepsilon \begin{bmatrix} \xi \\ \eta \end{bmatrix} + O(\varepsilon^2)$

Plug (2.55) into (2.54):

$X = x + \varepsilon \xi + O$

$Y = y + \varepsilon \eta + O$

$$\frac{\omega(x,y) + \varepsilon[\eta_x + \omega(x,y)\eta_y] + O}{1 + \varepsilon[\xi_x + \omega(x,y)\xi_y] + O} = \omega(x + \varepsilon\xi + O,\ y + \varepsilon\eta + O)$$

$$\frac{\varepsilon\eta_x + O + \omega(1 + \varepsilon\eta_y + O)}{1 + \varepsilon\xi_x + O + \omega(\varepsilon\xi_y + O)} \Leftarrow \begin{cases} X_x = 1 + \varepsilon\xi_x + O \\ X_y = \varepsilon\xi_y + O \\ Y_x = \varepsilon\eta_x + O \\ Y_y = 1 + \varepsilon\eta_y + O \end{cases}$$

$$= \frac{\omega + \varepsilon\eta_x + \varepsilon\omega\eta_y + O}{1 + \varepsilon(\xi_x + \omega\xi_y) + O}$$

$$h(\varepsilon) := \boxed{\frac{\omega + \varepsilon(\eta_x + \omega\eta_y) + O}{1 + \varepsilon(\xi_x + \omega\xi_y) + O}} = \overbrace{\omega(x + \varepsilon\xi + O,\ y + \varepsilon\eta + O)}^{g(\varepsilon)}$$

$h(0) + h'(0)\varepsilon + O$

$h'(\varepsilon) = \dfrac{lk' - kl'}{l^2} =$

$[1 + \varepsilon(\xi_x + \omega\xi_y) + O](\eta_x + \omega\eta_y + O(\varepsilon))$

$- [\omega + \varepsilon(\eta_x + \omega\eta_y) + O](\xi_x + \omega\xi_y + O(\omega))$

$g(0) + g'(0)\varepsilon + O$

$g'(\varepsilon) = D_1\omega\, \xi + D_2\omega\, \eta$

so $g'(0) = \omega_x(x,y)\xi + \omega_y(x,y)\eta$

$= \omega(x,y) + \omega_x\xi + \omega_y\eta$

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$$\frac{[1+\varepsilon(\xi_x+\omega\xi_y)+\Theta_2](\eta_x+\omega\eta_y+\Theta_1) - [\omega+\varepsilon(\eta_x+\omega\eta_y)+\Theta_1](\xi_x+\omega\xi_y+\Theta_1)}{[1+\varepsilon(\xi_x+\omega\xi_y)+\Theta_2]^2}$$

$$(1+\varepsilon(\xi_x+\omega\xi_y)+\Theta_2) = 1+2\varepsilon(\xi_x+\omega\xi_y)+\Theta_2$$

$$\eta_x+\omega\eta_y+\Theta_1+\cancel{\varepsilon(\xi_x+\omega\xi_y)(\eta_x+\omega\eta_y)}$$
$$-\omega(\xi_x+\omega\xi_y+\Theta_1)-\cancel{\varepsilon(\eta_x+\omega\eta_y)(\xi_x+\omega\xi_y)}$$

$$=\frac{(1-\omega)(\eta_x+\omega\eta_y+\Theta_1)+\Theta_2}{1+2\varepsilon(\xi_x+\omega\xi_y)+\Theta_2}\overset{!}{=}\omega_x\xi+\omega_y\eta$$

$$(1-\omega)(\eta_x+\omega\eta_y+\Theta_1)+\Theta_2=(\omega_x\xi+\omega_y\eta)(1+2\varepsilon(\xi_x+\omega\xi_y)+\Theta_2)$$
$$=\omega_x\xi+\omega_y\eta+2\varepsilon(\xi_x+\omega\xi_y)(\omega_x\xi+\omega_y\eta)+\Theta_2$$

<u>Want</u> $\eta_x+(\eta_y-\xi_x)\omega-\xi_y\omega^2=\xi\omega_x+\eta\omega_y$ (2.57)

$$\eta_x+\omega\eta_y-\omega\eta_x-\omega^2\eta_y$$

PLUG IN $\varepsilon=0$

$$\eta_x+\omega(\eta_y-\eta_x)-\omega^2\eta_y=\xi\omega_x+\eta\omega_y$$

It looks close, I must have made some algebra mistakes.

Skipping ahead & leaving this unfinished.

We know $\bar{Q}:=\eta-\omega\xi$

Then we can rewrite (2.57) as $\bar{Q}_x+\omega\bar{Q}_y-\omega_y\bar{Q}$ (2.58)

$$\eta_x-\omega_x\xi-\omega\xi_x+\omega(\eta_y-\omega_y\xi-\omega\xi_y)=\omega_y(\eta-\omega\xi)$$
$$\eta_x-\omega_x\xi-\omega\xi_x+\omega\eta_y-\cancel{\omega\omega_y\xi}-\omega^2\xi_y=\omega_y\eta-\cancel{\omega\omega_y\xi}$$
$$\boxed{\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y=\omega_x\xi+\omega_y\eta}$$

ok so (2.58) checks out.

p.18 A <u>trivial</u> Lie Symm has the Lie Sym flow along a soln curve to the ODE

$\bar{Q}=0\Leftrightarrow\eta=\xi\omega$

if $\bar{Q}$ satisfies (2.58) then $\begin{bmatrix}\xi\\ \eta\end{bmatrix}=\begin{bmatrix}\xi\\ \bar{Q}+\omega\xi\end{bmatrix}$ is a tangent vf of 1-param groups $\forall\ \xi$

In principle, non trivial syms can be found from (2.58) by method of characteristics.

ODE $\left(\frac{dx}{1}=\frac{dy}{\omega(x,y)}\right)=\frac{d\bar{Q}}{\omega_y(x,y)\,\bar{Q}}$ (2.59)