(3) (a) show $f: \mathbb{R} \to \mathbb{R}^3$, $t \mapsto \begin{bmatrix} t \\ t^2 \\ t^3 \end{bmatrix}$ embeds $\mathbb{R}$ in $\mathbb{R}^3$. To be an embedding:
(1) df_t one-to-one $\forall t$
(2) f is a homeo: f one-to-one ($\Rightarrow f^{-1}$ exists), f^{-1} cont

$$Df_t(h) = df_t(h) = \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix}[h] = h\begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix}$$

From 1st component, we can see this always one-to-one.

$df_0 = [1,0,0]^T$

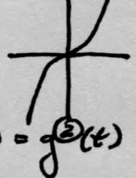
Likewise, from $f^{①}(t) = t$ we see f one-to-one and $f^{-1}(x,y,z) = x = f^{①}(x)$ smooth.

$\Rightarrow f$ is an embedding.

ASIDE: Contrast this with Avez DC p.116-117 $g: (-a, a) \to \mathbb{R}^2$, $t \mapsto \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$

$dg_t = \begin{bmatrix} 2t \\ 3t^2 \end{bmatrix}$ Here $dg_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so not one-to-one.

g is a homeo because $g^{②}(t) = t^3$



so we take $g^{-1}(x,y) = (g^{②})^{-1}(y) = y^{1/3}$

But this is not smooth: Let $F(t) = g^{②}(t) = t^3$. Then $(F^{-1})'(s) = \dfrac{1}{F'(t)}$ $F(t)=s$
$= \dfrac{1}{3t^2}$

$(F^{-1})'(0) = \frac{1}{0}$ undefined □

For ⓐ and fcn f

ⓑ Find 2 indep fcns $g = \begin{bmatrix} g^{①} \\ g^{②} \end{bmatrix} : \mathbb{R}^3 \to \mathbb{R}^2$ that carve out the image as $g^{-1}(0) = f(\mathbb{R})$. Are $g^{①}, g^{②}$ indep globally?

Let $g^{①}(x,y,z) = y - x^2$ then $g^{①}(t,t^2,t^3) = t^2 - t^2 = 0$
$g^{②}(x,y,z) = z - x^3$ $g^{②}(t,t^2,t^3) = t^3 - t^3 = 0$

$$Dg_{xyz} = \begin{bmatrix} -2x & 1 & 0 \\ -3x^2 & 0 & 1 \end{bmatrix}$$

These rows are LI globally (ie $\forall$ values x,y,z) □

(5) Given $f: \mathbb{R}^3 \to \mathbb{R}$, $x \mapsto x^2 + y^2 - z^2$

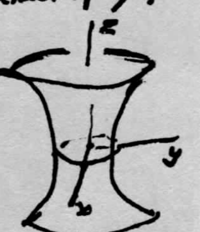
ⓐ show $0 \in \mathbb{R}$ is the only critical value.
[df_x maps onto $\mathbb{R}$ unless $f(x) = 0$]

ⓑ show $A := f^{-1}(a)$ and $B := f^{-1}(b)$ are diffeomorphic if a, b both pos OR both neg.

ⓒ Regard $c \in \mathbb{R}$ as a param. What happens to $S_c := f^{-1}(c)$ as c passes thru 0?
Bifurcation, Catastrophy.

ⓐ $df_x = Df_x = [2x \;\; 2y \;\; -2z]$
This maps onto $\mathbb{R}$ unless $\{0,0,0\}$ and $\{\{0,0,0\}\} = f^{-1}(0)$.

ⓑ $A = f^{-1}(a) = \{ \text{all } x \mid x^2 + y^2 - z^2 = a \}$ This is 1-sheet hyperboloid.



If $\alpha, \beta, \gamma > 0$ $\alpha x^2 + \beta y^2 - \gamma z^2 = 1$

$$\alpha x^2 + \beta y^2 = \underbrace{1 + \gamma z^2}_{\text{pos, call it } r_z^2}$$

Thus we have an ellipse for each z, and a 1-param family as z varies.

On the other hand, $\alpha x^2 - \beta y^2 - \gamma z^2 = 1$
$\alpha x^2 = 1 + \beta y^2 + \gamma z^2 \Rightarrow x = \pm \frac{1}{\sqrt{\alpha}} \sqrt{1 + \beta y^2 + \gamma z^2}$ H.B of 2 sheets.

(5) cont'd $\mathcal{A} := f^{-1}(a) = \{x^2+y^2-z^2=a\}$ $\quad$ $\mathcal{B} := f^{-1}(b) = \{x^2+y^2-z^2=b\}$

If $a>0$ $\frac{1}{a}x^2+\frac{1}{a}y^2-\frac{1}{a}z^2=1$ HB 1 sheet

If $b<0$ (write $-|b|$) $\Rightarrow \frac{-1}{|b|}x^2-\frac{1}{|b|}y^2+\frac{1}{|b|}z^2=1$ HB 2 sheets

Obviously there can be no diffeo between a 1-conn component and a 2-conn component pair of mfds.

If $a>0, b>0$, the diffeo is just a linear change of scale L:

$\frac{1}{a}x^2+\frac{1}{a}y^2-\frac{1}{a}z^2=1$ $\quad$ Let $\vec{x} = \frac{\sqrt{a}}{\sqrt{b}}\vec{u}$

$\Rightarrow \frac{1}{\cancel{a}}\frac{\cancel{a}}{b}u^2+\frac{1}{\cancel{a}}\frac{\cancel{a}}{b}v^2-\frac{1}{\cancel{a}}\frac{\cancel{a}}{b}w^2=1$

$\Rightarrow u^2+v^2-w^2=b$

Or, to rephrase that, let $\langle x,y,z\rangle \in \mathcal{A}$

that means $x^2+y^2-z^2=a$ ⊛

$L(x,y,z) := \frac{\sqrt{b}}{\sqrt{a}}$ [redacted] $\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \frac{\sqrt{b}}{\sqrt{a}}$ [redacted] $\begin{bmatrix}x\\y\\z\end{bmatrix}$ diffeo

Is $L(x,y,z)\in\mathcal{B}$? To see this, mult both sides of ⊛ by b/a [redacted]

$\frac{b}{a}$ [redacted] $x^2 + \frac{b}{a}$ [redacted] $y^2 - \frac{b}{a}$ [redacted] $z^2 = \frac{b}{\cancel{a}}\cancel{a}$

$= \left(\frac{\sqrt{b}}{\sqrt{a}}x\right)^2 + \left(\frac{\sqrt{b}}{\sqrt{a}}y\right)^2 - \left(\frac{\sqrt{b}}{\sqrt{a}}z^2\right) = b$

$\Rightarrow L(x,y,z)\in\mathcal{B} \Rightarrow L(A)=\mathcal{B}$

(really $\subseteq \mathcal{B}$ but I'm not doing any more).

(6) Let p be a homogeneous poly in k variables $x_1,\dots,x_k$. Homogeneous means $p(tx)=t^m p(x)$. That means each term must be of degree m

For example, if $k=3$ and $m=4$, $p(x) = x^4+x^2yz+xy^3+\dots$

(i) show $S_a = p^{-1}(a)$ is a $(k-1)$ dim submfd [$(k-1)$ dims get squashed]

(ii) show S_b diffeo to S_a if a,b both pos. [or both neg].

(i) $S_a = p^{-1}(a)$ that is $p(x)=a$ $\forall x\in S_a$

S_a is a mfd if a is a reg value of p, that is, dp_x maps onto $\mathbb{R}$ $\forall x\in S_a$

For a general p, it is hard to work with dp_x, but we can use Euler's homog thm: $x\cdot\nabla p(x) = m\,p(x)$

If $a\neq 0$ then $p(x)\neq 0 \Rightarrow x\cdot\nabla p(x)\neq 0$

observe that $v=\vec{0} \Rightarrow x\cdot v=0$

$v\neq\vec{0} \Leftarrow x\cdot v\neq 0$ contrapos

Thus $\nabla p(x)\neq\vec{0}$ i.e. $dp_x \neq \vec{0}^T$

So dp_x maps onto $\mathbb{R}$

$\Rightarrow S_a$ is a $(k-1)$ dim submfd of $\mathbb{R}^k$

(ii) I think we can define L just like in prob (5) $L(x) = \left(\frac{b}{a}\right)^{1/m}x$ $\quad$ $p(x) = \sum x_1^{r_1}x_2^{r_2}\dots x_k^{r_k} = a$ where $\sum^k r_i = m$

□

7 **Stack of Records Thm** (Some relationship to Covering Spaces in Munkres T ?)

$f: X \longrightarrow Y$ y Reg value
X cpt
$\dim X = \dim Y$

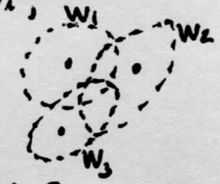
show $\Longrightarrow$

ⓐ $S = f^{-1}(y)$ is discrete set of pts

ⓑ $\exists$ a nbhd $V_y \ni f^{-1}(V_y) = \bigcup_{i=1}^{N} U_i$ where all U_i are disjoint and $f|_{U_i}: U_i \to V$ is a diffeo.

ⓐ $df_x : T_xX \longrightarrow T_yY$ map is Onto $\forall\, x \in f^{-1}(y) = S$
Since $\dim(T_xX) = \dim(T_yY)$ and finite, this shows df_x is nsing, thus an iso.
By Inv Fcn Thm, for each $x \in S$ $\exists$ nbhd U_x and $f|_{U_x}: U_x \to f(U_x)$ local diffeo.
The pts $x \in S$ are discrete because there can only be one in each nbhd U_x or else f not one-to-one there.
$\#S$ is finite because, if not, since X is cpt, any infinite set would have an accumulation pt and that would force many pre-image pts of y in a single U_x.

ⓑ Take $V_y = \bigcap^N f(U_i)$ open in Y since finite intersection.
Then $f^{-1}(V_y) = \bigcup^N W_i$, where each $W_i \subseteq U_i$ and $x_i \in W_i$
are the W_i disjoint?



Let $A_1 := W_1 - \overline{W_1 \cap W_2} - \overline{W_1 \cap W_3}$
$A_2 := W_2 - \overline{W_2 \cap W_1} - \overline{W_2 \cap W_3}$
$A_3 := W_3 - \overline{W_3 \cap W_1} - \overline{W_3 \cap W_2}$

Obviously A_i is disjoint from others and open. But does it contain x_i?
Let's just consider A_1 and say $N=2$. For x_1 to be taken out, it must be in $\overline{W_1 \cap W_2}$
That means a seq $(z_i) \to x_1$ and each $z_i \in W_2$. Since x_1 is an interior pt of W_1, all z_i for $i > M$ are in W_1. Then $f_{W_1}(z_i) \longrightarrow y$. But then $f_{W_2}^{-1}(f_{W_1}(z_i)) \longrightarrow x_2$ $\Rightarrow\Leftarrow$ since $x_2 \neq x_1$
Thus what I called "U_i" in the problem statement is really A_i
Maye take $V_y = \bigcap f(A_i)$ now. I don't care anymore.

▷ Now let's discuss Lie Groups
a Lie group is a manifold in $\mathbb{R}^N$ that also has a group structure.
Consider first $M(n)$ the set of all $n \times n$ matrices. It can be identified with $\mathbb{R}^{n^2}$ just by writing all the elts out in a long column, starting with matrix column 1.
But it is not a group because a matrix in $M(n)$ does not nec have an inverse.

Consider $GL(n) :=$ general linear group – all invertible maps $\mathcal{L}(\mathbb{R}^n \longrightarrow \mathbb{R}^n)$
$= \det^{-1}(\mathbb{R} - \{0\})$ so it is an open subset of $\mathbb{R}^{n^2}$ and thus a mfd.

$A \in GL(n)$
Tangent space $T_A(GL(n)) = \mathbb{R}^{n^2}$

Now we follow Avez DC p. 113, 119 Special Linear Group $SL(n)$ or Unimodular group in Avez terminology.

G&P Prob 11

$SL(n) := \{ \overset{\text{all}}{A} : \mathbb{R}^n \to \mathbb{R}^n \mid \det A = +1 \} = \det^{-1}(1)$

[$\det A = +1$ does not imply A is O.N.: $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$]

Let $f := \det$ then $f : \mathbb{R}^{n^2} \to \mathbb{R}$, $A \longmapsto \det(A)$

If we have that every $A \in SL(n)$ is a Reg value of f, then $SL(n)$ is a mfd.

This means showing $Df_A : \mathbb{R}^{n^2} \to \mathbb{R}$ is Onto (submersion) (i.e it is not 0)

We know from Avez ch 1 how to compute $Df_A(H)$, but to establish what we want here, we use a trick. Let $h(t) = tA$

$f(h(t)) = \underset{\text{LHS}}{f(tA)} = \underset{\text{RHS}}{t^n f(A)}$ since det is a homog poly

LHS: $\frac{d}{dt} f(h(t)) = Df_{h(t)}(h'(t)) = Df_{tA}(A)$

RHS: $\frac{d}{dt} t^n f(A) = n t^{n-1} f(A) \xrightarrow{\text{plug in } t=1}$ $Df_A(A) = n f(A) = n$ since $\det A = 1$

$\Rightarrow Df_A(A) \neq 0_{\mathbb{R}}$ so Df_A maps Onto $\mathbb{R}$.

$\Rightarrow \boxed{SL(n) \text{ is a mfd}}$ And $f : \mathbb{R}^{n^2} \to \mathbb{R}$ (n^2-1) dims are squashed

Thus $\boxed{SL(n) = f^{-1}(1) \text{ has dim } n^2 - 1}$

▷ Now show tangent space $T_I(SL(n))$ where I is identity matrix. Lie Algebra

We want to show $\boxed{T_I(SL(n)) = N \text{ where } N \text{ is the subsp of } \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n) \text{ where every } A \text{ has } Tr(A) = 0}$

Is this really a subsp? $\begin{matrix} Tr(A) = 0 \\ Tr(B) = 0 \end{matrix} \Rightarrow Tr(\alpha A + B) = 0$ ✓

Consider the curve $\gamma_A : \underset{\mathbb{R}}{(-\varepsilon, \varepsilon)} \to \mathbb{R}^{n^2}$, $t \longmapsto e^{tA}$ for some $A \in \cancel{SL(n)}\ N$

Since $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$ and we define $A^0 = I$, we have $\gamma(0) = I \in SL(n)$

For all t, $\gamma(t)$ stays in $SL(n)$ because of $\boxed{\text{Avez DC Thm 5.5} \quad \det(e^A) = e^{tr(A)}}$

so $\det(\gamma_A(t)) = \det(e^{tA}) = e^{tr(tA)} = e^{t\,Tr(A)} = e^{t\,0} = e^0 = 1$.

For a mfd M, we know $T_x M = \{ \overset{\text{all}}{\alpha'(0)} \mid \underset{\alpha(0) = x}{\alpha \text{ is a curve in } M} \}$

Now show $\gamma_A'(0) \in N$: $\boxed{\text{Avez Thm 5.6} \quad \gamma_A'(t) = A \gamma_A(t)}$ so $\gamma_A'(0) = A\gamma_A(0) = A \cdot I = A \in N$

$\Rightarrow T_I(SL(n)) \subseteq N$ and they have the same dim because $\dim(T_I(SL(n))) = \dim(SL(n)) = n^2 - 1$

But why is $\dim(N) = n^2 - 1$?

$N = \{ \overset{\text{all}}{n \times n} \text{ matrices with } Tr(A) = 0 \}$ all the entries of A are arb, except the main diag must satisfy $\sum^n a_{ii} = 0 \Rightarrow$ one entry is not arb e.g. $a_{nn} \stackrel{\perp}{=} \sum^{n-1} a_{ii}$

So only $n^2 - 1$ degrees of freedom

$N \cong \mathbb{R}^{n^2-1}$ $\dim(N) = n^2 - 1$ so $\dim(N) = \dim(T_I(SL(n)))$ and $T_I(SL(n)) \subseteq N$

$\Rightarrow \boxed{T_I(SL(n)) = N}$ □

▷ Now we want to consider the orthogonal group $\mathcal{O}(n)$ [It would be better to have called it O.N. group]

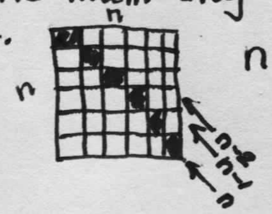
O.G. group $\mathcal{O}(n) := \left\{ \begin{matrix} \text{all} \\ Q \in M(n) \end{matrix} \;\middle|\; Q^TQ = I \right\}$ This is all matrices $Q: \mathbb{R}^n \to \mathbb{R}^n$ that preserve dist: (length) $\|Qx\|_2^2 = x^TQ^TQx = x^Tx = \|x\|_2^2$

We will show
(a) $\mathcal{O}(n)$ is submfd of $\mathbb{R}^{n^2}$ → $\dim(\mathcal{O}(n)) = \frac{n(n-1)}{2}$
(b) $\mathcal{O}(n)$ is cpt
(c) $\mathcal{O}(n)$ has 2 conn components
(d) Tangent space at Identity: $T_I(\mathcal{O}(n)) = \mathcal{A}$ ← subsp of skew-sym matrices: $A^T = -A$
[Remark: these also have $Tr(A)=0$]

(a) Define $f: M(n) \longrightarrow S(n)$ where $S(n)$ is the subsp of all sym matrices $A^T = A$
$A \longmapsto A^TA$

we will show $\mathcal{O}(n) = f^{-1}(I)$ and this is a mfd because I is a Regular value of f.

First, show $S(n)$ is a subsp, and it has dim $\frac{n(n+1)}{2}$:
obviously $(\alpha A+B)^T = \alpha A^T + B^T = \alpha A + B$ so subsp ✓
To establish dim, observe only the main diag and all elts above are free — the below ones are then determined.



$n + \sum_{i=1}^{n-1} i$ all squares $= n^2$

thus $2\sum_{i=1}^{n-1} i + n \overset{!}{=} n^2$

$\Rightarrow \sum_{i=1}^{n-1} i = \frac{n^2-n}{2} = \frac{n(n-1)}{2}$ This is the dim of $\mathcal{A}$

For the dim of $S(n)$: $n + \sum_{i=1}^{n-1} i = n + \frac{n(n-1)}{2} = \frac{2n+n^2-n}{2} = \frac{n(n+1)}{2}$

This is number of degrees of freedom

$S(n) \cong \mathbb{R}^{\frac{n(n+1)}{2}}$

▷ Now we must show Df_A maps Onto $T_{f(A)}S(n) = S(n)$ $\forall A \in f^{-1}(I)$

That is, for any $C \in S(n)$ does $\exists\ B \ni Df_A(B) = C$?

First we must compute Df_A:
$f(A) = A^TA$

$$f(A+H) - f(A) = (A+H)^T(A+H) - A^TA$$
$$= A^T(A+H) + H^T(A+H) - A^TA = \cancel{A^TA} + A^TH + H^TA - \cancel{A^TA}$$

Thus we see $Df_A(H) = A^TH + H^TA$ and then we want $A^TH + H^TA \overset{!}{=} C$ where $C = C^T$

Take $H := \frac{1}{2}AC \Rightarrow \frac{1}{2}\left(A^TAC + (AC)^TA\right) = \frac{1}{2}\left(IC + \underset{C}{\underset{\shortparallel}{C^T}}\underbrace{A^TA}_{I}\right) = C$

So Df_A maps Onto $T_I(S(n))$ and thus $\mathcal{O}(n) = f^{-1}(I)$ is a mfd.
What is $\dim(\mathcal{O}(n))$? How many dimensions are squashed? $n^2 - \frac{n(n+1)}{2} = \boxed{\frac{n(n-1)}{2}}$

(b) To show $\mathcal{O}(n)$ is cpt, we show it is clsd and bdd:
since $Q^TQ = I$ $\vec{q}_i \cdot \vec{q}_i = 1$ $\|\vec{q}_i\|_2^2 = 1 \Rightarrow |q_{ij}| \le 1$ for all j and for each i ⇒ Bdd.
$\mathcal{O}(n) = f^{-1}(I)$ and I is a single pt in $\mathbb{R}^{\frac{n(n-1)}{2}}$. Thus $\{I\}$ is clsd and f is cont ⇒ $f^{-1}(I)$ is clsd.

cont'd ⟶

(c) $O(n)$ consists of 2 Conn components

$O(n) = \{ \text{all } Q \mid Q^TQ = I \}$ apply det: $\det(Q^TQ) = \det I$, $\det Q^T \det Q = 1 \Rightarrow (\det Q)^2 = 1 \Rightarrow \det Q = \pm 1$

For a cont fcn g, g(conn set) = conn set

det is a cont fcn and $\det(O(n)) = \{-1\} \cup \{1\}$ 2 disjoint sets, disconnected

Thus we see $O(n) = \det^{-1}(\{-1\}) \cup \underbrace{\det^{-1}(\{1\})}_{SO(n)}$ These must be disjoint sets

since $\det I = +1$, we distinguish this component as $SO(n)$ or the Rotation group (these O.N. maps Q do no inversions, only $\mathcal{O}$ preserving rotations).

NOTE from wikipedia:

$SO(2) \cong S^1$

$SO(3) \cong \mathbb{RP}^3$

(d) Lie algebra for $O(n)$ Avez DC p.119

Show $T_I(O(n)) = \mathcal{A}$ where $\mathcal{A} := \{ \text{all anti-sym maps } A \mid A^T = -A \}$

observe this means $\langle Ax, x \rangle = 0$ because $x^TAx = x^TA^Tx = x^T(-A)x = -x^TAx$

From discussion for part (a), we established $\dim \mathcal{A} = \frac{n(n-1)}{2}$

we want to find $T_I(O(n))$. we will do this by finding an arb curve γ in $O(n)$ and computing its tangent vector $\gamma'(0)$.

Choose any $A \in \mathcal{A}$, we want to show that $\gamma(t) = e^{tA}$ is contained in $O(n)$.

This means $\langle e^{tA}, e^{tA} \rangle = (e^{tA})^T e^{tA} = I$. alternately, we can show $Q := e^{tA}$ satisfies $\|Qx\| = \|x\| \ \forall x \in \mathbb{R}^n$. $\boxed{\text{show } \|Qx\|_2^2 = x^TQ^TQx = \|x\|_2^2}$

Choose arb $x \in \mathbb{R}^n$

$\gamma(t) = Q_t = e^{tA}$ and we use whichever form is most suggestive.

We define $F(t) := \langle e^{tA}x, e^{tA}x \rangle \equiv \langle \gamma(t)x, \gamma(t)x \rangle \equiv \langle Q_tx, Q_tx \rangle$

$$\boxed{\text{If } F'(t) = 0 \text{ then } \underbrace{F(t)}_{= \|Qx\|^2} = \text{const} = F(0) = \langle Ix, Ix \rangle = \|x\|^2}$$

$F'(t) = \langle \gamma'(t)x, \gamma(t)x \rangle + \langle \gamma(t)x, \gamma'(t)x \rangle = 2 \langle \gamma(t)x, \gamma'(t)x \rangle$

From $\boxed{\text{Avez DC Thm 5.6}}$ $\gamma'(t) = Ae^{tA}$ Thus $F'(t) = 2\langle e^{tA}x, Ae^{tA}x \rangle = 2\langle Qx, AQx \rangle$

But $F'(t) = 0$ because $A^T = -A$:

$\langle Qx, AQx \rangle = (AQx)^TQx = x^TQ^T\underbrace{A^T}_{-A}Qx = -x^TQ^T(AQx) = -\langle Qx, AQx \rangle \Rightarrow \langle Qx, AQx \rangle = 0$

$\Rightarrow F(t) = F(0) \ \forall t$

Thus we know $e^{tA} \in O(n) \ \forall t \Rightarrow \gamma(t)$ is a curve in $O(n)$

Again from Avez Thm 5.6 $\gamma'(0) = Ae^{0A} = AI = A$

So $A \in T_I(O(n)) \Rightarrow \mathcal{A} \subseteq T_I(O(n))$

Now we observe $\dim(\mathcal{A}) = \frac{n(n-1)}{2} = \dim O(n) = \dim T_I(O(n))$

$\Rightarrow \boxed{T_I(O(n)) = \mathcal{A}}$ $\boxed{\text{QED}}$

Avez DC p.121 How could we compute $T_Q(\mathcal{O}(n))$ and $T_U(SL(n))$ for a base pt that is not I?

$\mathcal{L}(\mathbb{R}^n \to \mathbb{R}^n) \cong \mathbb{R}^{n^2}$ as before. For $A \in GL(n)$ (invertible) define "left operator"

$$L_A : \mathcal{L} \longrightarrow \mathcal{L}$$
$$G \longmapsto AG$$

we found $T_I(SL(n)) = \mathcal{N}$ = Trace 0 matrices because we chose $A \in \mathcal{N}$ and we showed

(1) $\cdot\ \gamma(t) = e^{tA} \in SL(n)\ \forall t\ \left[\det e^{tA} = 1\right]$

(2) $\cdot\ \gamma'(0) = A \in \mathcal{N}$

(3) $\cdot\ \gamma(0) = I$

Thus $T_I(SL(n)) \subseteq \mathcal{N}$ and they have same dim $\Rightarrow \subsetneq$

Here we need to show:

(1) For arb $U \in SL(n)$, $L_U(\gamma(t)) = (U \circ \gamma)(t) \in SL(n)$

If $C \in SL(n)$ then $\det C = +1$ and $\det(UC) = \det U \cdot \det C = 1 \cdot 1 = 1$

and we know $\gamma(t) = C = e^{tA}$

So in fact we have $L_U : SL(n) \longrightarrow SL(n)$ Linear Iso (in fact, diffeo)

U invertible, $\det U = 1$

$L_U(A+B) = U(A+B) = UA + UB = L_U A + L_U B$

$L_{U^{-1}} = (L_U)^{-1}$: $L_U L_{U^{-1}}(A) = UU^{-1}A = A$ and $L_{U^{-1}} L_U(A) = U^{-1}UA = A$

(2) $(U \circ \gamma)'(0) = DU_{\gamma(0)}(\gamma'(0)) \overset{U \text{ linear}}{=} UA \in U(\mathcal{N}) = L_U(\mathcal{N})$

(3) $(U \circ \gamma)(0) = U(\gamma(0)) = U(I) = U$

$\Rightarrow$ Thus we have $T_U(SL(n)) \subseteq L_U(\mathcal{N})$ and since L_U is an iso, the dims are same so we have equality.

$$\boxed{T_U(SL(n)) = L_U(\mathcal{N})}$$

$\triangleright$ Now show $\boxed{T_Q(\mathcal{O}(n)) = L_Q(\mathcal{A})}$ recall $T_I(\mathcal{O}(n)) = \mathcal{A} = \left\{\begin{matrix}\text{anti-symm} \\ -A = A^T\end{matrix}\right\}$

choose arb $Q \in \mathcal{O}(n)$ again $L_Q : \mathcal{O}(n) \longrightarrow \mathcal{O}(n)$ is a linear iso ($\Rightarrow$ diffeo)

$$R \longmapsto QR$$

$(QR)^T QR = R^T Q^T QR = I$ ✓

Choose $A \in \mathcal{A}$ then we know $\gamma(t) = e^{tA} \in \mathcal{O}(n)$

(1) $(Q \circ \gamma)(t) = Q \cdot e^{tA} \in \mathcal{O}(n)$

(2) $(Q \circ \gamma)'(0) = DQ(\gamma'(0)) = QA \in Q(\mathcal{A}) = L_Q(\mathcal{A})$

(3) $(Q \circ \gamma)(0) = Q e^{0A} = QI = Q \Rightarrow T_Q(\mathcal{O}(n)) = L_Q(\mathcal{A})$

QED