G&P ch 1.3 and 1.4 — **Inv Fcn Thm, Immersions, Submersions** — 7/6/2023

p.13 For smooth maps f, the behavior of df_x dominates the behavior of f in a nbhd of x.

Thm $f: X \to Y$ diffeo $\Rightarrow$ df_x is an iso for all $x \in X$

Pf.
$$\begin{tikzcd} U \arrow[r, "f"] & V \\ \mathcal{U} \arrow[u, "\varphi"] \arrow[r, "h"] & \mathcal{V} \arrow[u, "\psi"'] \end{tikzcd}$$
Since f is a diffeo, we can trim down U if nec so $f(U) \subseteq \psi(\mathcal{V})$

Then $h: \mathcal{U} \to \mathcal{V}$ is a diffeo in $\mathbb{R}^k$ and $df_x = d\psi \cdot Dh \cdot d\varphi^{-1}$ – each term is an iso □

Inv Fcn Thm on Mfds $\exists\, \underset{\sim}{x} \ni df_{\underset{\sim}{x}}$ is an iso $\Rightarrow$ f is a local diffeo in nbhd of $\underset{\sim}{x}$

Pf. We know $df_{\underset{\sim}{x}} = d\psi \cdot Dh \cdot d\varphi^{-1}$ and a priori $d\psi$ and $d\varphi$ are isos.

Since df_x is an iso $\Rightarrow$ Dh_x must be also.

Apply ord calculus $\mathbb{R}^k$ Inv Fcn Thm and see $\exists$ nbhd $\mathcal{A}$ of x and $\mathcal{B}$ of $h(x)$ where $h: \mathcal{A} \to \mathcal{B}$ is a diffeo $\Rightarrow f$ is a diffeo from $\varphi(\mathcal{A}) \xrightarrow{f} \psi(\mathcal{B})$ □

We say 2 maps f and $\bar{f}$ are '<u>equivalent</u> up to diffeo' if $\exists$ diffeos $\alpha, \beta \ni$ the following diagram commutes (we also say f & $\bar{f}$ are <u>conjugate</u>)

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ \bar{X} \arrow[r, "\bar{f}"] & \bar{Y} \end{tikzcd}$$

Thus if f is a diffeo, we can say f is <u>Conj</u> to Id:

Really charts here U, V

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\ \mathbb{R}^k \arrow[r, "Id"] & \mathbb{R}^k \end{tikzcd}$$

because

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\alpha"'] & V & \psi \arrow[l] \\ \mathcal{U} \arrow[r, "Id"] & \mathcal{U} \arrow[r, "h"] & \mathcal{V} \arrow[u] \end{tikzcd}$$

Just call this ψ

▷ Let's contrast this with <u>Immersions</u> and <u>Submersions</u>:

If $\dim X \neq \dim Y$ no diffeos possible. But we can have <u>immersions</u> (and possibly <u>embeddings</u>) when $\dim Y$ is bigger than that of X

we can have <u>Submersions</u> (and possibly pre-image mfds) when $\dim Y$ smaller than X.

I am combining local immersions (ch 1.3) and local submersions (ch 1.4) because the proofs share many similiarities.

Local Immer Thm $f: X^{(k)} \to Y^{(\ell)}$ $k<\ell$, $\exists$ pt $\underset{\sim}{x} \ni df_{\underset{\sim}{x}}: T_xX \to T_{f(x)}Y$ is <u>One-to-One</u> $\Rightarrow$ $\exists$ charts such that

$$\begin{tikzcd} U^{(k)} \arrow[r, "f"] & V^{(\ell)} \\ \mathcal{U}^{(k)} \arrow[u, "\varphi"] \arrow[r, "i"] & (\mathcal{U} \times \{0\})^{(\ell)} \arrow[u, "\psi"'] \end{tikzcd}$$

where $i = \begin{array}{c} k \\ \left[\begin{array}{c} I \\ \hline 0 \end{array}\right] \ell \end{array}$ is the Cannonical Immersion

Local Sub Thm $f: X^{(k)} \to Y^{(\ell)}$ $k>\ell$, $\exists$ pt $\underset{\sim}{x} \ni df_x$ is <u>Onto</u> $\Rightarrow$ $\exists$ charts so that

$$\begin{tikzcd} U^{(k)} \arrow[r, "f"] & V^{(\ell)} \\ \mathcal{U}^{(k)} \arrow[u, "\varphi"] \arrow[r, "\pi"] & \mathcal{V} \arrow[u, "\psi"'] \end{tikzcd}$$

where $\pi = \overbrace{\left[\begin{array}{c|c} I & 0 \end{array}\right]}^{k} \ell$ is Cannonical Sub

NOTE: $\pi^T = i$

cont'd →

For the pfs, we first need the following lemmas using a change of basis (or change of variables) to bring a linear map to canonical form:

Linear Sub Lemma: matrix $A: \mathbb{R}^k \to \mathbb{R}^\ell$ $k > \ell$, A is Onto $\mathbb{R}^\ell$ $\Big\} \Rightarrow$ $\exists$ nsing matrix $B \ni$ $\boxed{AB = \pi}$

i.e.

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "A"] & \mathbb{R}^\ell \arrow[d, leftrightarrow, "I"] \\ \mathbb{R}^k \arrow[u, "B"] \arrow[r, "\pi"] & \mathbb{R}^\ell \end{tikzcd}$$

Pf. Since A is Onto, $\exists\, v_1, .., v_\ell \in \mathbb{R}^k \ni Av_i = e_i \in \mathbb{R}^\ell$

Is $\{v_1 \dots v_\ell\}$ LI? Yes, because if we suppose not: $\sum^\ell \alpha_i v_i = 0$ for some $\alpha_i \neq 0$

Apply A: $A(\sum \alpha_i v_i) = A0 = 0$

$\sum \alpha_i A v_i$

$\sum^\ell \alpha_i e_i \overset{!}{=} 0 \Rightarrow\Leftarrow$ obviously the std basis $\{e_i\}$ is LI in $\mathbb{R}^\ell$.

By Rank + Nullity thm, $\dim(\ker A) = k - \ell$ and there are LI vectors $\{v_{\ell+1}, \dots v_k\}$ which span $\ker(A)$

Then $\{v_1 \dots v_\ell, v_{\ell+1} \dots v_k\}$ is a basis for $\mathbb{R}^k$ satisfying

$$Av_i = \begin{cases} e_i & 1 \le i \le \ell \\ 0 & i > \ell \end{cases}$$

$\Rightarrow$ The matrix B is $\begin{bmatrix} | & & | \\ v_1 & \dots & v_k \\ | & & | \end{bmatrix}$ It is nsing and $AB = \pi = [I \;\vdots\; 0]$ $\square$

Using this, we can prove:

Linear Immer Lemma: matrix $A: \mathbb{R}^k \to \mathbb{R}^\ell$ $k < \ell$, A is One-to-One $\Big\} \Rightarrow$ $\exists$ nsing matrix $C: \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $\boxed{CA = \imath}$

i.e.

$$\begin{tikzcd} \mathbb{R}^k \arrow[r, "A"] \arrow[d, "I"] & \mathbb{R}^\ell \arrow[d, "C"] \\ \mathbb{R}^k \arrow[r, "\imath"] & \mathbb{R}^k \times \{0\} \end{tikzcd}$$

Pf. $A: \mathbb{R}^k \longrightarrow \mathbb{R}^\ell$ has rank k

$\mathbb{R}^k \xleftarrow{A^T} \mathbb{R}^\ell$ also has rank k (row rank = col rank)

$\Rightarrow$ A^T maps onto $\mathbb{R}^k$ so it is a linear submersion.

By Linear Sub lemma $\exists\, B \ni A^T B = \pi : \mathbb{R}^\ell \longrightarrow \mathbb{R}^k$

Transpose everything: $(A^T B)^T = \pi^T : \mathbb{R}^\ell \longleftarrow \mathbb{R}^k$

$B^T A$ rename this as C and we are done! $\square$

Repeating

Local Immer thm $f: X^{(k)} \to Y^{(\ell)}$ $k < \ell$, $\exists x$ $df_x: T_xX \to T_yY$ is One-to-One $\Big\} \Rightarrow$ $\exists$ charts such that

$$\begin{tikzcd} U^{(k)} \arrow[r, "f"] & V^{(\ell)} \\ \mathcal{U}^k \arrow[u, "\varphi"] \arrow[r, "\imath"] & (\mathcal{U} \times \{0\})^\ell \arrow[u, "\psi"] \end{tikzcd}$$

$(\mathcal{U} \times \{0\}) \subset \mathcal{V}^\ell$ thickening

domain of ψ is a thickening of the

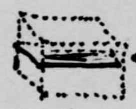
COR: f is immer at $x \Rightarrow f$ immer in nbhd of x

Pf. It is tricky and hard to motivate beforehand.

A priori we have

$$\begin{tikzcd} U \arrow[r, "f"] & V \\ \mathcal{U} \arrow[u, "\varphi"] \arrow[r, "h"] \arrow[dr, dashed, "\tilde{h}"] \arrow[ddr, "\imath"'] & \mathcal{V} \arrow[u, "\psi"] \arrow[d, shift left, "C"] \\ & \mathcal{W} \arrow[u, shift left, "C^{-1}"] \\ & (\mathcal{U} \times \{0\}) \arrow[u, "G"] \end{tikzcd}$$

we need to create all this



NOTE

$\mathcal{U} \times \{0\}$ in a open set in $\mathbb{R}^\ell$ G

step 1 we need to express h as $h = G \circ \imath$ for some map G which is a diffeo

Then $Dh_x = DG_{\imath(x)}\, \imath$

$$\begin{array}{c} \\ \ell \end{array} \begin{bmatrix} | & | & | \\ & Dh_x & \\ | & | & | \end{bmatrix}_{\ell \times k} = \ell \left[\begin{array}{c:c} \overset{k}{A} & S \\ & \ell \end{array} \right] \begin{bmatrix} I \\ \cdots \\ 0 \end{bmatrix}_{\ell}^{k}$$

Thus we see block $A \overset{!}{=} Dh_x$ and S can be anything since it is killed by $\imath$

cont'd $\rightarrow$

Step 2 we will choose $S = \begin{array}{c} \ell-k \\ \begin{bmatrix} 0 \\ I \end{bmatrix} \end{array}\begin{array}{l} k \\ \ell-k \end{array}$ and our first attempt at G will be

$$G : \mathbb{R}^k \times \mathbb{R}^{\ell-k} \longrightarrow \mathbb{R}^\ell$$
$$(x, z) \longmapsto \begin{bmatrix} h(x) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ z \end{bmatrix}$$

Then we get $DG_{xz} = \begin{array}{cc} k & \ell-k \\ \left[\begin{array}{c:c} Dh_x & 0 \\ \hdashline & I \end{array}\right] \end{array} k$

we need to show this is nsing, because then we can apply Inv Fcn Thm and establish G is local diffeo.

Step 3 we could do this by making a linear COV on $\mathbb{R}^\ell$

By <u>Linear Immersion Lemma</u> $\exists \; C : \mathbb{R}^\ell \to \mathbb{R}^\ell$ nsing and $CA = \imath'$ $(A = Dh_x$ here$)$

Now define $\hbar := C \circ h$ and redefine $G(x,z) = \begin{bmatrix} \hbar(x) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ z \end{bmatrix} = Ch(x) + \begin{bmatrix} 0 \\ z \end{bmatrix}$

Now $DG_{xz} = \left[\begin{array}{c:c} CA & 0 \\ & I \end{array}\right] = \left[\begin{array}{c:c} I & 0 \\ \hdashline 0 & I \end{array}\right]$ obviously this is nsing.

Step 4 apply Inv Fcn Thm and get that G is a diffeo on a <u>nbhd</u> in $\mathcal{U} \times \{0\}$ (in fact, in a thickening of $\mathcal{U} \times \{0\}$)

Define new $\bar{\psi} = \psi \circ C^{-1} \circ G$ on a trimmed down $\hat{\mathcal{U}} \times \{0\}$ in $\mathcal{U} \times \{0\}$ and we are done.

pf. $f = (\psi \circ C^{-1} \circ G) \circ \imath \circ \varphi^{-1}$
- φ^{-1} is a diffeo
- $\imath : \mathcal{U} \to \mathcal{U} \times \{0\}$ is a diffeo
- $\psi \circ C^{-1} \circ G$ is a local diffeo □

COR: f local Immer $\Rightarrow$ f local diffeo onto its <u>image</u>

Recall

<u>Local Sub Thm</u> $f : X^{(k)} \to Y^{(\ell)}$ $k > \ell$, $\exists x \ni df_x$ maps Onto T_yY $\Big\} \Longrightarrow$ $\exists$ charts such that

$$\begin{tikzcd} U \arrow[r, "f"] & V \\ \mathcal{U} \arrow[u, "\varphi"] \arrow[r, "\pi"] & \mathcal{V} \arrow[u, "\psi"'] \end{tikzcd}$$

COR

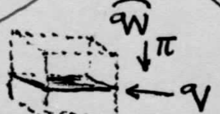
f sub at $x \Rightarrow f$ sub in nbhd of x

The Preimage Thm is an illustrative COR to Loc Sub Thm — see sheet (14)

pf. again we have the std setup, and we need to create new maps to shoehorn in π

$$\begin{tikzcd} U \arrow[r, "f"] & V \\ \mathcal{U} \arrow[u, "\varphi"] \arrow[r, "h"] & \mathcal{V} \arrow[u, "\psi"'] \\ \mathcal{W} \arrow[u, "B"] \arrow[ur, dotted, "\hbar"] \arrow[d, "G"] & \\ \hat{\mathcal{W}} \arrow[uur, "\pi"'] & \end{tikzcd}$$

$\mathcal{A} \xrightarrow{G} \mathcal{B}$



(π collapses $\mathcal{W}$ onto $\mathcal{V}$)

Step 1 Now we need $h = \pi \circ G$ for some diffeo G.

This implies $Dh_x = \underbrace{\begin{array}{cc} \ell & k-\ell \\ \left[\begin{array}{c:c} I & 0 \end{array}\right] \end{array}}_{\pi} \begin{array}{c} k \\ \left[\begin{array}{c} A \\ \hdashline S \end{array}\right] \end{array}\begin{array}{l} \ell \\ k-\ell \end{array}$ $\quad$ S can be arb since it is crushed by π.

Step 2 we will take $S = [\, 0 \,\vdots\, I \,]$

and first def of G is $G : \mathbb{R}^k \longrightarrow \mathbb{R}^k$

$$x \longmapsto \begin{bmatrix} h(x) \\ x^{\ell+1} \\ \vdots \\ x^{k} \end{bmatrix}$$

Then $DG_x = \left[\begin{array}{c:c} \multicolumn{2}{c}{Dh_x} \\ \hdashline 0 & I \end{array}\right]$

Step 3 To show DG_x is nsing, we need to make a linear COV

From <u>Linear Sub Lemma</u> $\exists$ nsing $B : \mathbb{R}^k \to \mathbb{R}^k$ such that $AB = \pi$ $(A = Dh_x)$

Define $\hbar = h \circ B : \underbrace{B^{-1}(\mathcal{U})}_{\mathcal{W}} \longrightarrow \mathcal{V}$ and redine $G(x) = \begin{bmatrix} \hbar(x) \\ x^{\ell+1} \\ \vdots \\ x^{k} \end{bmatrix}$

$DG_x = \left[\begin{array}{c:c} \multicolumn{2}{c}{Dh \cdot B} \\ \hdashline 0 & I \end{array}\right] = \left[\begin{array}{c:c} I & \\ \hdashline & I \end{array}\right]$ obviously nsing.

Step 4 Apply Inv Fcn Thm and we get that G is a local diffeo on nbhds $\mathcal{A} \to \mathcal{B}$. Trim down the size of the sets $\mathcal{U}, \mathcal{V}$. Redefine φ as $\varphi \circ B \circ G^{-1} : \mathcal{B} \to \hat{U}$ and $\psi : \pi(\mathcal{B}) \longrightarrow \hat{V}$ smaller

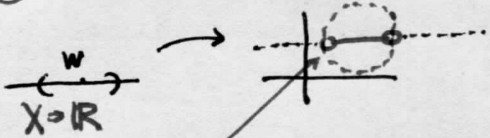
QED

The image of $\dot{\imath}: \mathbb{R}^k \longrightarrow \mathbb{R}^\ell$ is the nicest example of a submfd

p.13 Let's consider an arb imersion $f: X \longrightarrow Y$

The book says "for $f(X)$ to be a mfd, pts must have parameterizable nbhds, but the subsets $f(W)$ [W open set in X] need not be open in Y" Error: should be $f(X)$.

To see this is an error, consider $\dot{\imath}: \mathbb{R} \hookrightarrow \mathbb{R}^2$



$f(W)$ is not an open set in $Y = \mathbb{R}^2$. But it is $f(X) \cap \Delta$ where Δ is an open disc in $\mathbb{R}^2$ $\Rightarrow$ open set in $f(X)$

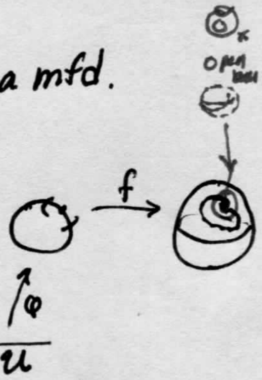
Re-summarizing from earlier in this chapter:
The set X is a k-dim submfd of $\mathbb{R}^N$ if every pt $x \in X$ is contained in an open nbhd $U_x \subset X$ that is diffeomorphic (via chart map) to an open set in $\mathbb{R}^k$: $\varphi: \mathcal{U} \longrightarrow U_x$

X inherits the topology of $\mathbb{R}^N$, so any open set in X is of the form $U_x = X \cap \mathbb{U}_x$ where $\mathbb{U}_x$ is an open set in $\mathbb{R}^N$.
We can work "in the small" and say $\mathbb{U}_x = \mathring{B}(x, \varepsilon_x)$. Every pt $x \in X$ is an interior pt so $x \in U_x := X \cap \mathring{B}(x, \varepsilon_x)$ [No mfd-w/-∂ here]

Thm (implicitly stated by G&P)

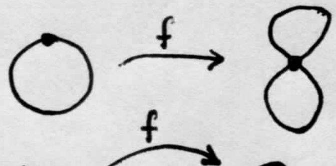
f one-to-one?
- For any open set $W \subset X$, $f(W)$ open in $f(X)$
- f is a local diffeo onto its image (we get this from Loc Immer Thm Cor)

$\Bigg\} \Rightarrow f(X)$ is a mfd.

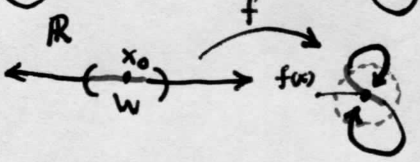


Pf. Take $W = U_x$, a tiny open nbhd around x
$f(U_x)$ is open in $f(X)$ by hypoth $\Rightarrow f(U_x) = f(X) \cap \mathbb{O}_{f(x)}$
Since f is a local diffeo onto its image, by shrinking $f(U_x)$ if nec, we have $(f \circ \varphi): \mathcal{U} \longrightarrow f(U_x)$ as a chart for any $f(x) \in f(X)$ $\square$

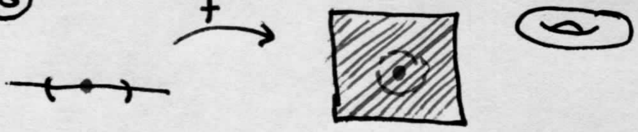
▷ Let's look at some pathalogical immersions where this doesn't hold [Note: these immersions may still be useful mathematical objects even if not mfds — see 'chains' in Spivak CoM

① 

Self intersection - f not One-to-One

② 

As $|x| \to \infty$ in $\mathbb{R}$, these pts asymptotically approach $f(x_0)$ in the image
No open nbhd in $f(X)$ contains only pts in $f(W)$, no chart possible.

③ 

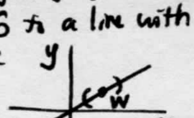
Torus $S^1 \times S^1$ represented as a flat square.
A line winds densely around the torus, so no chart can be formed.

G&P give more details on the construction

$g: \mathbb{R} \to S^1$
$t \longmapsto \begin{bmatrix} \cos 2\pi t \\ \sin 2\pi t \end{bmatrix}$

$G: \mathbb{R}^2 \to S^1 \times S^1$
$(x, y) \longmapsto (g(x), g(y))$

Now restrict G to a line with irrational slope



Recall f cont $\Leftrightarrow f^{-1}(\text{open}) = \text{open}$
also $f(\text{cpt}) = \text{cpt}$

<u>Def</u> $f: X \to Y$ is <u>Proper</u> if $f^{-1}(C)$ is cpt whenever C is cpt subset of Y.

<u>Def</u> $f: X \to Y$ is an <u>embedding</u> if (1) f is an immersion [df_x One-to-One $\forall x \in X$]
(2) f is One-to-One [so f^{-1} exists]
(3) f is Proper [This means f^{-1} cont]

i.e. • df_x is always One-to-One
• f is a global homeo

Thm $f: X \to Y$ is embedding $\Rightarrow \begin{cases} f: X \to f(X) \subseteq Y \text{ is a global diffeo} \\ f(X) \text{ submfd of } Y \end{cases}$

Pf. First we show $f(X)$ is a mfd. From the 'implicitly stated thm' on prev sheet, it is enough to show: For any open set $W \subseteq X$, $f(W)$ is open in $f(X)$.

<u>Step 1</u> Suppose $f(W)$ <u>not</u> open in $f(X)$
Suppose pt $y \in f(W)$ that is not an interior pt
Then $\exists$ at least one pt $y \in f(W)$ that is not an interior pt $y_i \in f(X)$ and $y_i \notin f(W)$
$\Rightarrow y$ is a limit pt of some seq (y_i) where each $y_i \in f(X)$ and $y_i \notin f(W)$ with only 1 lim pt, namely y
By my design, (y_i) is a bdd, discrete set with only 1 lim pt, namely y
Thus $S := (y_i)_{i=1}^{\infty} \cup \{y\}$ is a cpt set

<u>Step 2</u> Since f is Proper, $f^{-1}(S)$ is a cpt set in X
Since f is One-to-one, $\exists!$ $x_i = f^{-1}(y_i)$ and $x = f^{-1}(y)$. So we have seq (x_i)

<u>Step 3</u> Does $(x_i) \to x$?
We know $(x_i) \cup \{x\}$ is cpt $\Rightarrow$ its seq cpt: every seq has a convergent subseq (x_{i_k})
Say $(x_{i_k}) \to z$ By Continuity

$$\lim_k f(x_{i_k}) = f(\lim x_{i_k}) = f(z)$$
$$\lim y_{i_k} = y \Rightarrow f(z) = y$$
f One-to-One $\Rightarrow z = x$
$\Rightarrow (x_{i_k}) \to x$

<u>Step 4</u> $y \in f(W)$
$\rightarrow x \in W$ an open set
Than means x is an interior pt, so since $(x_{i_k}) \to x$,
all $x_{i_k} \in W$ for k large enough $\Rightarrow y_{i_k} \in f(W)$ but this is a contradiction

<u>Step 5</u> So every pt $y \in f(W)$ must be an interior pt
$\Rightarrow f(W)$ is open in $f(X)$
$\Rightarrow f(X)$ is a submfd of Y

<u>Step 6</u> Now we must show $X \xrightarrow{f} f(X)$ is a global diffeo
We know f is local diffeo, and f is globally One-to-One
f^{-1} exists and is smooth $\Rightarrow f$ global diffeo □

<u>Claim</u>: When X is cpt, every f that is a <u>One-to-One</u> <u>Immersion</u> is an <u>embedding</u> (we get Proper for free).
Pf. Let $K \subseteq Y$ be cpt. Then K clsd, bdd. $f^{-1}(K)$ is clsd in X just because f is Cont. $f^{-1}(K) \cap X$ is thus cpt
[cf Rudin POMA p.37] □

The Pre-image Thm (or Reg value thm) is the most important Cor of Local Sub Thm. The pf is hard to understand without an illustrative picture, given below.

Def $y \in Y$ is a regular value for f if df_x maps Onto $T_{f(x)}Y \ \forall\, x \in f^{-1}(y)$

If y is not a reg value, we call it a critical value.

P.21 "Absurd Punctillio" – If $y \notin f(X)$ we call it a regular value still. Gompf says "Nothing to check"

Preimage Thm (Cor of Loc Sub Thm)

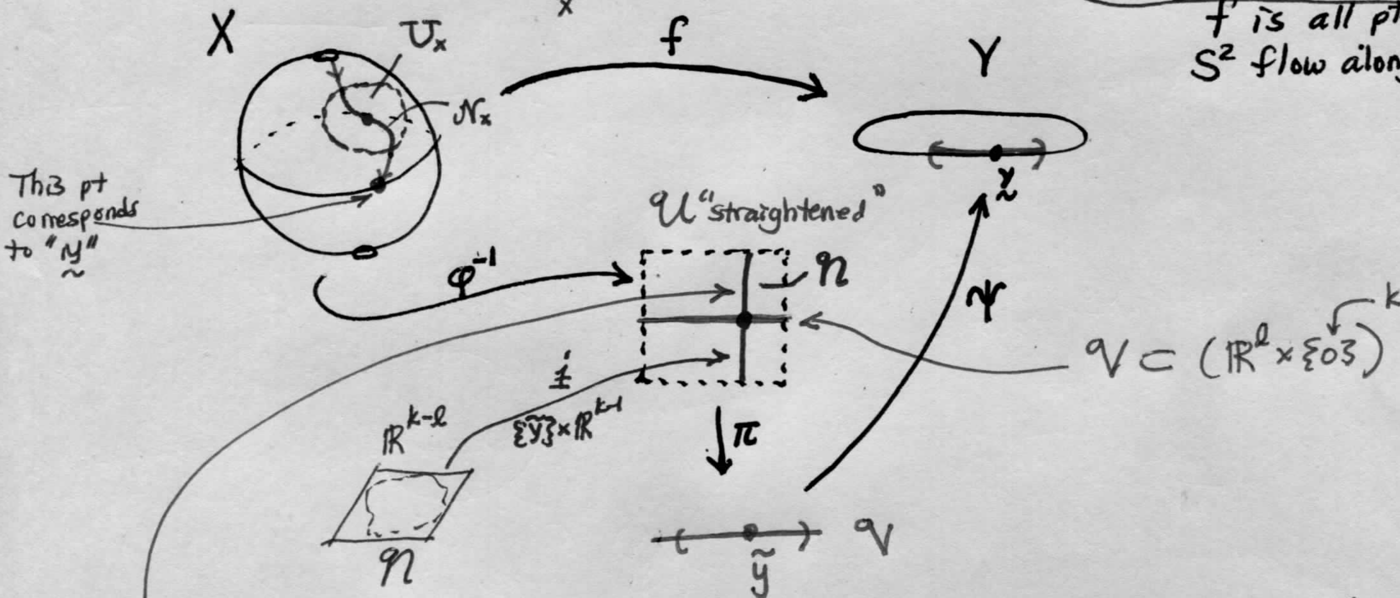
$$\left.\begin{array}{l} f: X^{(k)} \longrightarrow Y^{(\ell)} \quad k \geq \ell \\ \underset{\sim}{y} \text{ is a reg value of } f \ [df_x \text{ Onto } \forall x \in f^{-1}(\underset{\sim}{y})] \end{array}\right\} \Longrightarrow \begin{array}{l} ① \ Z = f^{-1}(\underset{\sim}{y}) \text{ is submfd of } X \\ ② \ \dim(Z) = \dim X - \dim Y \end{array}$$

"$\text{codim}_X(Z)$" $= \ell$ $= (k-\ell)$

The dimension of Z is the number of dims that got squashed in the submersion

Pf. The Loc Sub Thm shows each pt $\underset{\sim}{z}_x \in f^{-1}(\underset{\sim}{y})$ has a chart:

f is all pts on punctured S^2 flow along curves to equator.



This "straight fibre" is collapsed onto $\{\tilde{y}\}$ by π. It is an open "interval" in $(k-\ell)$ dim hyperplane. That is to say, $\mathcal{N}$ is the inclusion into $\mathbb{R}^\ell \times \mathbb{R}^{k-\ell}$ of an open set in $\mathbb{R}^{k-\ell}$.

Thus $\varphi|_{\mathcal{N}}$ or $\varphi \circ \mathbb{1}: \overset{\mathbb{R}^{k-\ell}}{\mathcal{N}} \longrightarrow N_x$ is a chart for the open nbhd of $f^{-1}(\underset{\sim}{y})$ contained in U_x [The set $U_x \cap N_x$ is open in $f^{-1}(\underset{\sim}{y})$] so N_x is a submfd.

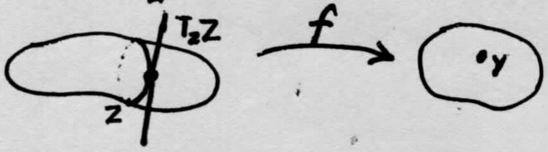
If we can do this for all pts $\underline{x} \in f^{-1}(\underset{\sim}{y})$ [df_x is Onto $\forall x$] $\Rightarrow Z = f^{-1}(\underset{\sim}{y})$ is a submfd.

And since Z is locally diffeomorphic to $\mathbb{R}^{k-\ell}$, its dimension is $(k-\ell)$ $\boxed{\text{QED}}$

I am going out-of-seq with G&P and presenting the rest of the theorems before Lie Group examples.

p.24 **Prop 3** $f: X \to Y$, Submfd $Z = f^{-1}(y)$ where y is reg value $\Bigg\} \Rightarrow \ker(df_z) = T_zZ \quad \forall z \in Z$

Pf. Step 1 show $T_zZ \subseteq \ker(df_z)$:



since $f: Z \to \{y\}$ (const.), we have $df_z(v) = 0 \quad \forall v \in T_zZ$

why?

$$\begin{tikzcd} Z \arrow[r, "f"] & \{y\} \\ \mathcal{U} \arrow[u, "\varphi"] \arrow[r, "c"] & \{a\} \arrow[u, "\psi"] \end{tikzcd}$$

$df_z(v) = d\psi \circ Dc \circ d\varphi^{-1}$ and we know from calculus $DC_z = 0$

Thus $df_z(v) = 0 \quad \forall v \in T_zZ$

Step 2 Now show $\dim[\ker(df_z)] = \dim[T_zZ]$

Rank + Nullity Thm

$\dim[\ker(df_z)] = \dim[\text{domain of } df_z] - \dim[\operatorname{Im}(df_z)]$

$= \dim T_zX - \dim T_{f(z)}Y = \dim X - \dim Y \underset{\uparrow \text{ Preimage Thm}}{=} \dim Z = \dim T_zZ$ $\square$

$\Rightarrow \ker(df_z) = T_zZ$

▷ We can reformulate the Pre-Image Thm in terms of a submfd being 'cut out' by 'independent fcns':

Consider $g: X \to \mathbb{R}^\ell$, $x \mapsto \begin{bmatrix} g^{1}(x) \\ \vdots \\ g^{\ell}(x) \end{bmatrix}$

For 0 to be a Reg value, we need $dg_x: T_xX \to \mathbb{R}^\ell$ Onto $\forall x \in Z = f^{-1}(0)$

A := $dg_x = \begin{bmatrix} - dg^1 - \\ \vdots \\ - dg^\ell - \end{bmatrix}$

Claim dg_x maps Onto $\mathbb{R}^\ell \iff$ The fcnals $\{dg^{1}_x, \dots, dg^{\ell}_x\}$ are LI

($\Rightarrow$) $\dim(A(T_xX)) = \ell$ so A has rank ℓ and row rank = col rank
we have ℓ rows dg^{i} and they must be LI to have rank ℓ.

($\Leftarrow$) The ℓ rows of matrix A are LI so A has rank $\ell \Rightarrow$ col vectors span subsp of dim ℓ
The image $A(T_xX) \subseteq \mathbb{R}^\ell$ and $\mathbb{R}^\ell$ has dim ℓ so $A(T_xX)$ must be all of $\mathbb{R}^\ell \Rightarrow A$ maps Onto $\mathbb{R}^\ell$ $\square$

Define $\operatorname{Codim}_X(Z) = \dim X - \dim Z$

Def: We say the fcns $\{g^{i}\}$ are Indep at x if $\{dg^i_x\}$ is LI.

Def If $Z = g^{-1}(0)$ and $\{g^{i}\}$ is indep $\forall z \in Z$ we say Z is cut out by indep fcns.

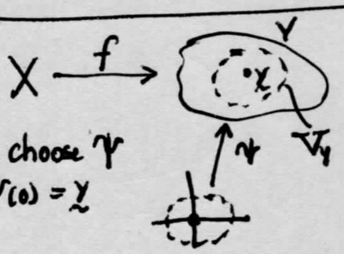
Can every Submfd $Z \subseteq X$ be cut out by indep fcns?

This is equivalent to saying: 'is every submfd a Preimage submfd?'

Generally: NO, not globally anyway. cf. prob #20 ch 2.3

Partial Result 1 Z is a submfd from Preimage Thm $\Rightarrow Z = g^{-1}(0)$ cut out by indep fcns
[That is, $Z = f^{-1}(y)$ for reg value]

Pf



We choose ψ so $\psi(0) = y$

we can define $g := \psi^{-1} \circ f : f^{-1}(V_y) \to \mathbb{R}^\ell$ — we know $f^{-1}(V_y)$ covers Z

we then know $g(z) = 0 \quad \forall z \in Z$

we know $dg_z = \underbrace{d\psi^{-1}}_{\text{iso}} \circ df_z$ maps Onto $\mathbb{R}^\ell$

$\square$

$k > \ell$

Partial Result 2 $Z^{(\ell)}$ arb submfd of $X^{(k)} \Longrightarrow$ Locally Z can be cut out by indep fcns

pf is ch 1.3 prob #2: Since Z is a submfd $\quad Z^{(\ell)} \overset{\mathbb{R}^N}{} \xhookrightarrow{i} X^{(k)} \overset{\mathbb{R}^N}{}$ I would say this is an embedding.

We can apply Loc Immer Thm if di_z is One-to-one.

Easy: since $i := Id_{\mathbb{R}^N}\big|_Z : Z \hookrightarrow X \subseteq \mathbb{R}^N$

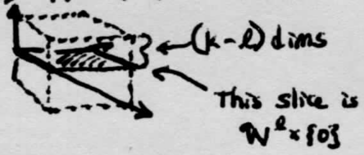
$$di_z = DI_z\big|_{T_zZ} = I\big|_{T_zZ}$$ obviously I is One-to-One.

Then

$$\begin{tikzcd} W_z^{(\ell)} \arrow[r, "f = i"] \arrow[dr, bend left, "G"] & X^{(k)} \\ \mathcal{W}^{(\ell)} \arrow[u, "\varphi"] \arrow[r, "i"] & \mathcal{W}^{\ell} \times \{0\}^{k-\ell} \subset \mathbb{R}^k \arrow[u, "\psi"] \end{tikzcd}$$

$\leftarrow$ should be U

$\boxed{\text{LIT}}$



Define $G : W_z \longrightarrow \mathbb{R}^k$
$$w \longmapsto \psi^{-1} \circ f(w) = \begin{bmatrix} * \\ * \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \}\ \ell \\ \\ \}\ k-\ell \end{matrix}$$

So we see $\left.\begin{aligned} G^{(\ell+1)} &= 0 \\ &\vdots \\ G^{(k)} &= 0 \end{aligned}\right\}$ on W_z

These are the indep fcns we seek

Let $g : W_z \longrightarrow \mathbb{R}^{k-\ell}$
$$w \longmapsto \begin{bmatrix} G^{\ell+1} \\ \vdots \\ G^{k} \end{bmatrix} \qquad W_z = g^{-1}(0)$$

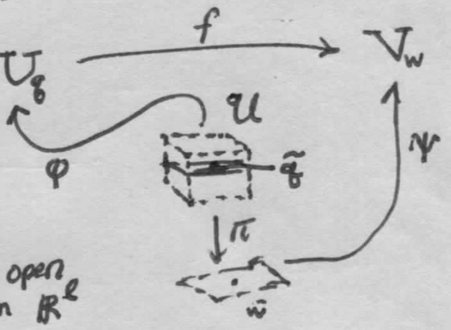
$\boxed{\text{QED}}$

Let's do the problems NOT related to Lie groups first:

ch 1.4 ① $f : X \longrightarrow Y$ submersion, $\mathcal{O}$ open set in X $\Longrightarrow f(\mathcal{O})$ open in Y.

Choose any $w \in f(\mathcal{O})$ and show it has a nbhd contained in $f(\mathcal{O})$

Let $f(q) = w$. Then by Loc Sub Thm, $\exists$ charts



By shrinking U if nec, $q \in U \subseteq \mathcal{O}$

Since canonical proj π preserves open sets $\pi(\mathcal{U}) = \mathcal{V}$ open in $\mathbb{R}^\ell$

$\Rightarrow w \in \psi(\pi(\mathcal{U})) \subseteq f(\mathcal{O})$ so w is an interior pt $\square$

② ⓐ $\left.\begin{aligned} &f : X \to Y \text{ submersion} \\ &X \text{ cpt},\ Y \text{ conn} \end{aligned}\right\} \Rightarrow$ f maps X onto all of Y, $f(X) = Y$

Pf. From ①, we know $f(X)$ is open in Y (since X open in X). But since X cpt, f cont $\Rightarrow f(X)$ cpt $\Rightarrow$ clsd, Bdd.
$\Rightarrow f(X)$ is both Open and Clsd in Y. From the upcoming lemma, this means $f(X) = Y$ $\square$

Lemma: Set Y is conn $\Rightarrow$ only Y and $\emptyset$ are both Open and Clsd in Y

pf. Let $\mathcal{a} \subseteq Y$ and $\mathcal{a}$ open and clsd. Since $\mathcal{a}$ clsd, $B := Y \setminus \mathcal{a}$ is open $\Rightarrow Y = \mathcal{a} \cup B$, $\mathcal{a} \cap B = \emptyset \Rightarrow Y$ has been disconn $\Rightarrow\!\Leftarrow$

ⓑ $f : X \longrightarrow \mathbb{R}^n$, X cpt $\Rightarrow f$ cannot be a submersion

This is a cor of ⓐ: If f would be a Sub, then $f(X) = \mathbb{R}^n$ but $f(X)$ is cpt and $\mathbb{R}^n$ is not $\Rightarrow\!\Leftarrow$ $\square$