G&P Ch 1 Summary

3/21/2016

Ch 1.1

The main goal of this chapter is to show that, for a smooth map $f: X \to Y$, the local behaviour is entirely specified (up to diffeo) by $df_x: T_xX \to T_{f(x)}Y$ p.13

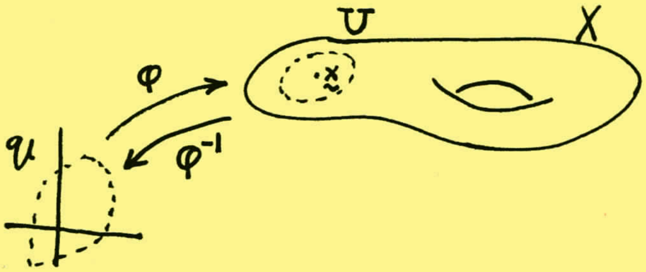
From calculus, we know that a map $F: \underset{\mathbb{R}^n}{\mathcal{U}} \to \mathbb{R}^m$ is <u>smooth</u> if all its partial derivs $\dfrac{\partial F^i}{\partial x_j}$ are smooth.

For a map $f: \underset{\mathbb{R}^n}{X} \to \mathbb{R}^m$ where X is not an open set in $\mathbb{R}^n$ (say X is sphere S^2 in $\mathbb{R}^3$) we define f to be <u>smooth</u> if: for each $x \in X$, $\exists$ an open set $\mathbb{U}$ in $\mathbb{R}^n$ and a map with smooth partials $F: \mathbb{U} \to \mathbb{R}^m \ni F\big|_{\mathbb{U} \cap X} = f$

Let $\mathcal{A}$ be a subset of $\mathbb{R}^p$ and $\mathcal{B}$ subset of $\mathbb{R}^q$

$f: \mathcal{A} \to \mathcal{B}$ is a <u>diffeo</u> if $\begin{cases} f \text{ smooth } \forall x \in \mathcal{A} \\ f \text{ One-to-one} \\ f \text{ Onto} \\ f^{-1} \text{ exists and is smooth at all } b \in \mathcal{B} \end{cases}$

$X \subset \mathbb{R}^N$ is a $\boxed{\text{k-dim mfd}}$ if every $\underset{\sim}{x} \in X$ has a nbhd $U_{\underset{\sim}{x}}$ (open in X, so it is $\mathbb{U}_{\underset{\sim}{x}} \cap X$ for open set $\mathbb{U}$) that is diffeo to an open set $\mathcal{U} \subseteq \mathbb{R}^k$

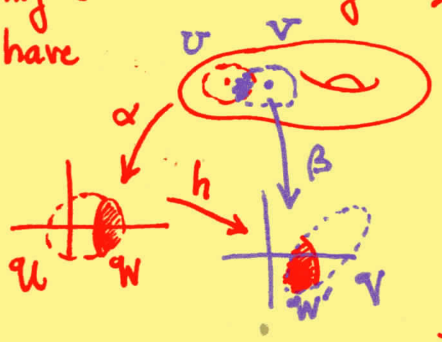


The map $\varphi: \mathcal{U} \to U$ is called a <u>parameterization</u> of a patch of the mfd.

The map $\varphi^{-1}: U \to \mathcal{U}$ is called a <u>local co-ord system</u> and the components, $(\varphi^{-1})^{i} = x^{i}$ are called <u>co-ord fcns</u>.

φ^{-1} is smooth in the previous sense of each $\underset{\sim}{x} \in X$ having a nbhd $\mathbb{U}_{\underset{\sim}{x}}$ where φ^{-1} extends to a classically smooth fcn $\Phi: \mathbb{U}_x \to \mathcal{U}$

In the abstract setting (Loomis & Sternberg AC), there is <u>no reference</u> to an embedding space $\mathbb{R}^N$. Instead we have



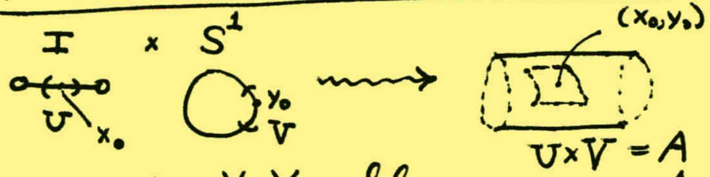
(Note the map α points the other way).

The mfd is declared <u>smooth</u> if when 2 charts have overlap $W = U \cap V$, then there is a diffeomorphism $h: \mathcal{W} \to \mathcal{W}'$ (transition fcn)

G&P make no mention of this, since for them, the maps α and β would already be diffeos, so $h = \beta \circ \alpha^{-1}$ is automatically a diffeo.

Thm we can make new mfds from existing mfds by the cartesian product.

$$\left.\begin{array}{l} X \text{ k-dim mfd in } \mathbb{R}^N \\ Y \text{ l-dim mfd in } \mathbb{R}^M \end{array}\right\} \Rightarrow X \times Y \text{ is a } (k+l)\text{-dim mfd in } \mathbb{R}^{N+M}$$



pf Fix $(x_0,y_0) \in X\times Y$. Show $\exists$ nbhd A which is open in $X\times Y$ and a diffeo $\theta: \mathcal{A} \to A$.

Define $\theta := \varphi \times \psi$ $\quad A := U\times V$ and $\mathcal{A} := \mathcal{U}\times\mathcal{V}$

▷ $\varphi\times\psi: \mathcal{U}\times\mathcal{V} \to U\times V \subset \mathbb{R}^N\times\mathbb{R}^M$ clearly this map is One-to-One and Onto because φ and ψ are separately and they don't interfere with each other.

$$(x_1,\dots,x_k,y_1,\dots,y_l) \longmapsto \begin{bmatrix}\varphi(x)\\ \psi(y)\end{bmatrix}$$

▷ $D(\varphi\times\psi)_{(x,y)} = \begin{bmatrix} D\varphi_x & \\ & D\psi_y \end{bmatrix}$ obviously every component partial is smooth, so $D(\varphi\times\psi)$ is.

▷ Is $(\varphi\times\psi)^{-1}$ smooth? $\quad (\varphi\times\psi)^{-1} = \varphi^{-1}\times\psi^{-1}$

we know φ^{-1} is smooth: For any $x\in X$ $\exists$ $\mathbb{U}_x$ and $\tilde{\varphi}^{-1}: \mathbb{U}_x \to \mathbb{R}^k$

ψ^{-1} is smooth: $y \in Y$ $\exists$ $\mathbb{V}_y$ $\quad \tilde{\psi}^{-1}: \mathbb{V}_y \to \mathbb{R}^l$

$\mathbb{U}\times\mathbb{V}$ is open in $\mathbb{R}^{M+N}$ and $\tilde{\varphi}^{-1}\times\tilde{\psi}^{-1}: \mathbb{U}\times\mathbb{V} \to \mathbb{R}^k\times\mathbb{R}^l$

$$\varphi^{-1}\times\psi^{-1} = \tilde{\varphi}^{-1}\times\tilde{\psi}^{-1}\Big|_{(\mathbb{U}\times\mathbb{V})\cap X\times Y}$$ so it is smooth. $\square$

Problems

(6) Show that a map which is smooth, One-to-One and Onto is still not nec a diffeo:

Consider $f: \mathbb{R} \to \mathbb{R}$, $x \longmapsto x^3$ $\qquad f^{-1}(y) = y^{1/3}$ fails to be dif'b at 0

Let $g = f^{-1}$. $g'(y) = \frac{1}{3}y^{-2/3} = \frac{1}{3}\frac{1}{y^{2/3}}$ undefined for $y=0$

observe $f'(x) = 3x^2$, $f'(0) = 0$

[Remark: we could also do this with $g'(y) = \frac{1}{f'(x)} = \frac{1}{3x^2} = \frac{1}{3(y^{1/3})^2}$]

(7) Are the x and y co-ord axes a mfd in $\mathbb{R}^2$?

No. Any pt away from origin has a nbhd trivially diffeomorphic to $\mathbb{R}^1$

Any nbhd of the origin is an open disk in $\mathbb{R}^2$ intersected with axes $\Rightarrow$ call this X

This is not even homeo to $\mathbb{R}$. Suppose it were: $\varphi: I \to X$

If we remove the origin from X, we have 4 conn components, but if we remove $\varphi^{-1}(0,0)$ from interval I, we are removing 1 pt from an Interval which would leave 2 conn components. The image of a connected set is connected under a cont map, so φ can't be cont $\Rightarrow\Leftarrow$

$\square$

(11) Show that the sphere S^k cannot be parameterized by a single chart $\varphi: \mathcal{U} \to S^k$.

By def, $\mathcal{U}$ is an open set. S^k is cpt. φ^{-1} must be smooth $\Rightarrow \varphi^{-1}$ cont

If h is a cont map, $h(\text{cpt}) = \text{cpt}$ so $\varphi^{-1}(S^k) = \mathcal{U}$ should be cpt, but $\mathcal{U}$ is open

Heine-Borel Thm states cpt sets in $\mathbb{R}^k$ are clsd and bdd $\Rightarrow\Leftarrow$ well, $\mathcal{U}$ could be $\varnothing$ but that is itself a contradiction. $\square$

Ch 1.2 Derivatives and Tangents

<u>Remark</u> G&P severely overuse the symbol 'd'.

For $f: \mathbb{R}^n \to \mathbb{R}^m$ we have $\delta f(x;h) = \lim_{t\to 0} \frac{1}{t}\left[f(x+th) - f(x)\right]$ Gateaux deriv

$$Df_x(h) = \begin{bmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \frac{\partial f^m}{\partial x_1} & & \frac{\partial f^m}{\partial x_n} \end{bmatrix}\begin{bmatrix} h \end{bmatrix}$$ Frechet deriv

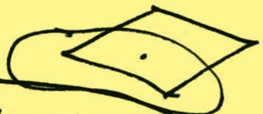
They call both of these $df_x(h)$ as well as the map to be introduced $df_x : T_xX \to T_yY$

<u>Tangent Space</u> For a chart $\varphi: \underset{\mathbb{R}^k}{\mathcal{U}} \to \underset{\mathbb{R}^N}{X}$ and G&P say $\varphi(0) = x$, but this is unnec.

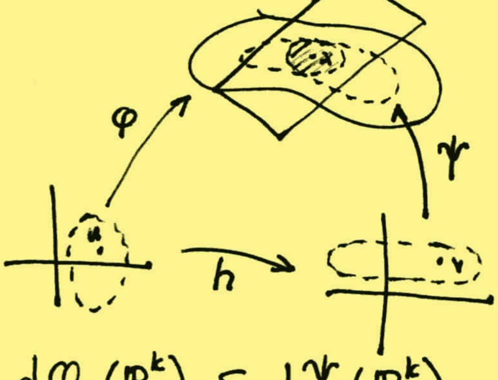
we have $D\varphi_0 : \mathbb{R}^k \to \mathbb{R}^N$ $\quad D\varphi_0 = \begin{array}{c} k \\ \left[\frac{\partial \varphi^i}{\partial x_j}\right]_N \end{array}$ They call this $d\varphi_0$

$$\boxed{\text{Define } T_xX = D\varphi_0(\mathbb{R}^k)}$$

This is really a hyperplane at the origin but we imagine it translated to touch mfd



$$\boxed{\text{Does } T_xX \text{ depend on the particular param } \varphi: \mathcal{U} \to U?}$$



$\varphi = \Psi \circ h$

$d\varphi_u = d\Psi_{h(u)} \circ dh_u$

I'm going to write dh_u as Dh_u since it is the familiar calculus map $Dh_u : \mathbb{R}^k \to \mathbb{R}^k$ and it is an <u>iso</u>

Need to show $d\varphi_u(\mathbb{R}^k) \subseteq d\Psi_v(\mathbb{R}^k)$ and vice-versa.

($\subseteq$) choose $\xi \in T_xX$

Then $\exists\, \xi \in \mathbb{R}^k \ni d\varphi_u(\xi) = \underset{\sim}{\xi}$ but $d\varphi_u = d\Psi_{h(u)} \circ dh_u$ so $\underset{\sim}{\xi} = d\Psi_{h(u)}(dh_u(\xi))$

($\supseteq$) Likewise if $\underset{\sim}{\eta} \in d\Psi_v(\mathbb{R}^k)$, $\underset{\sim}{\eta} = d\varphi_u(Dh_v^{-1}(\eta))$ because Dh_u is an iso

$\square$

p.9 Show $\dim(T_xX) = k$ when X is a k-dim mfd

pf Since x is in a chart, we know $\varphi^{-1}: U \to \mathcal{U}$ exists and is smooth
i.e. $\exists$ an open set $\mathbb{U}_x$ and a map $\Phi: \mathbb{U}_x \to \mathbb{R}^k \ni \varphi^{-1} = \Phi|_{\mathbb{U} \cap X}$
Thus $\Phi \circ \varphi = Id_{\mathbb{R}^k} \Rightarrow D\Phi_{\varphi(x)} \circ d\varphi_u = I$

$$\begin{bmatrix} & & \\ & & \end{bmatrix}\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} I \end{bmatrix}$$

This implies that $d\varphi_u$ must be One-to-One $\Rightarrow$ The k cols of $d\varphi_u$ are LI
$\Rightarrow$ They span a k-dim subsp in $\mathbb{R}^N$
$\Rightarrow \dim(T_xX) = k$ $\square$

$d\varphi_u$ is an iso onto its image T_xX.

suggests "T" for tangent map.

▷ For a map $f: X \to Y$, define $df_x : T_xX \to T_{f(x)}Y$ $\quad y = f(x)$

Given
$$\begin{tikzcd} U \arrow[r, "f"] & V \\ \mathcal{U} \arrow[u, "\varphi"] \arrow[r, "h"] & \mathcal{V} \arrow[u, "\psi"'] \end{tikzcd}$$
we may have to shrink $\mathcal{U}$ to do this.

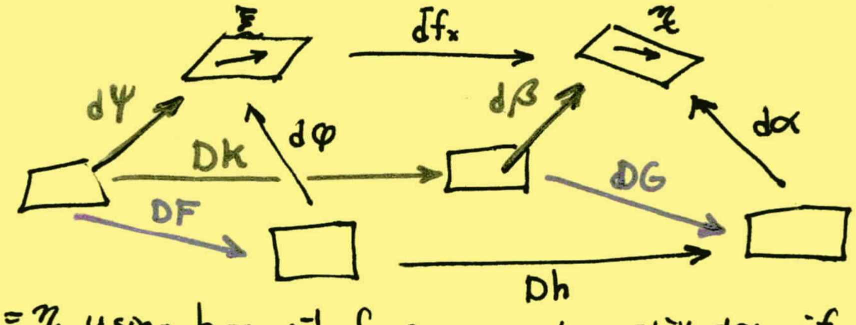
$h := \psi^{-1} \circ f \circ \varphi$

Define
$$\begin{tikzcd} T_xX \arrow[r, "df_x"] & T_{f(x)}Y \\ \mathbb{R}^k \arrow[u, "d\varphi_u"] \arrow[r, "Dh_v"] & \mathbb{R}^l \arrow[u, "d\psi_v"'] \end{tikzcd}$$

Then define $df_x(\cdot) := d\psi_v \circ Dh_u \circ d\varphi_u^{-1}(\cdot)$

we require that:
- df_x is linear: $df_x(\lambda\xi + \eta) = \lambda df_x(\xi) + df_x(\eta)$
- For $X = \mathbb{R}^k$, $Y = \mathbb{R}^l$ this should be the familiar Df_x
- Chain rule holds $X \xrightarrow{f} Y \xrightarrow{g} Z$ $\quad d(g \circ f) = dg_y \cdot df_x$

obviously df_x is linear, because the downstairs route is.
If $d\varphi_u = Id_{\mathbb{R}^k}$ and $d\psi = Id_{\mathbb{R}^l}$ then $df_x = Df_x$
Verify that df_x does not depend on the charts (φ, U) and (ψ, V):



If $df_x(\xi) = \eta$ using $h = \alpha^{-1} \circ f \circ \varphi$, is this still true if we use $k = \beta^{-1} \circ f \circ \psi$?
Yes because on the overlap of the co-ord patches we must have transition fcns $\psi = \varphi \circ F$ and $\beta = \alpha \circ G$ $\quad [\psi^{-1} = F^{-1} \circ \varphi^{-1}]$
Then by the chain rule from Calculus, $df_x(\xi) = d\beta \circ DK \circ d\psi^{-1}(\xi)$

$$= d\alpha \circ \underbrace{DG \circ DG^{-1}}_{I} \circ Dh \circ \underbrace{DF \cdot DF^{-1}}_{I} \circ d\varphi^{-1}(\xi)$$
$$= d\alpha \circ Dh \circ d\varphi^{-1}(\xi)$$
$$= \eta \quad \square$$

The chain rule for tangent maps

$$\begin{tikzcd} U \arrow[r, "f"] & V \arrow[r, "g"] & W \\ \mathcal{U} \arrow[u, "\varphi"] \arrow[r, "h"'] & \mathcal{V} \arrow[u, "\psi"] \arrow[r, "k"'] & \mathcal{W} \arrow[u, "\lambda"'] \end{tikzcd}$$

want: $d(g\circ f)_x = dg_{f(x)} \cdot df_x$

we know $d(g\circ f) = d\lambda \cdot \underbrace{D(k\circ h)}_{Dk\cdot Dh} \cdot d\varphi^{-1}$

$$= \underbrace{d\lambda \cdot Dk \cdot d\psi^{-1}}_{dg_y} \cdot \underbrace{d\psi \cdot Dh \cdot d\varphi^{-1}}_{df_x} \qquad \square$$

Let's do some illustrative problems:

Ch 1.2 #1 Inclusion map Let X submfd Y, then $i : X \hookrightarrow Y$

This strange map is just the identity $I : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ where the first $\mathbb{R}^N$ has only X and the 2nd has Y (containing X)

Show: $di_x : T_xX \longrightarrow T_xY$ is inclusion of T_xX into T_xY

we have

$$\begin{tikzcd} \underset{\mathbb{R}^N}{X} \arrow[r, "i"] & \underset{\mathbb{R}^N}{Y} \\ \mathcal{U} \arrow[u, "\varphi"] \arrow[r, "h"] & \mathcal{V} \arrow[u, "\psi"'] \end{tikzcd}$$

but we don't know what h is. But here we can avoid the downstairs route: $i = I$ in $\mathbb{R}^N$

$di_x = DI_x = I$

Ch 1.1 #5

Let V be a k dim vector space in $\mathbb{R}^N$ (hyperplane thru origin). Show V is diffeo to $\mathbb{R}^k$ and thus a mfd.

Let V have basis $\{v_1 \dots v_k\}$. Then any $v \in V$ is $v = \sum^k x_i \vec{v}_i$

$\Rightarrow$ the global diffeo is $\varphi : \mathbb{R}^k \longrightarrow V$

$$(x_1 \dots x_k) \longmapsto \sum^k x_i \vec{v}_i = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$$

linear map! $\square$

Ch 1.2 #3 Show $T_xV = V$ for V as in prev problem.

By def, $T_xV = d\varphi_u(\mathbb{R}^k)$ $\quad \varphi(u) = x$

but since φ is the linear map A, $d\varphi_u = A$ also

$T_xV = A(\mathbb{R}^k) = V$ $\square$

$\Rightarrow T_x\mathbb{R}^n = \mathbb{R}^n$

(Ch 1.2 #4) Show $f: X \to Y$ diffeo $\Rightarrow df_x : T_xX \to T_{f(x)}Y$ iso (6)

(The converse will be Inv Fcn Thm on mfds)

For any pt $x \in X$, we have charts which make the following diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \varphi\uparrow & & \uparrow\psi \\ \mathcal{U} & \xrightarrow{h} & \mathcal{V} \end{array}$$

$h := \psi^{-1} \circ f \circ \varphi$ is a diffeo

Dh_u is an iso (because h has an inverse $g \ni h \circ g = I$)

$D(h \circ g) = I$

$Dh_u \cdot Dg_{h(u)} = I$

$Dg_v = [Dh_u]^{-1}$

$\Downarrow$

$df_x = d\psi \cdot Dh_u \cdot d\varphi_x^{-1}$ is then an iso $\square$

(Ch 1.2 #5) Show there is no diffeo $\psi: \mathbb{R}^k \to \mathbb{R}^l$ for $k \neq l$

If there would be a diffeo ψ, then $d\psi_x = D\psi_x$ would be an iso, by previous problem.

But as vector spaces $\dim(\mathbb{R}^k) = k$ and $\dim(\mathbb{R}^l) = l$ so there can be no iso between them $\Rightarrow\Leftarrow$

Compare this arg to Spivak:

(Spivak CON 2.37) (a) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be C^1 smooth. Show f can't be One-to-One.

First we show $\exists$ an open set where one partial deriv (wlog $D_1 f = f_x$) is not zero.

Suppose $\exists$ an open ball $\mathring{B}(x, \varepsilon)$ where $Df_x = [0 \ 0]$

Then by MVT $f(x) - f(y) = Df_\xi (y - x) = 0 \Rightarrow f(x) = f(y)$ for $x, y \in B(x, \varepsilon)$

$\Rightarrow\Leftarrow$ Not One-to-One

$\triangleright$ So we see we must have an open, dense set B where $Df_x \neq 0$ meaning $f_x \neq 0$ or $f_y \neq 0$

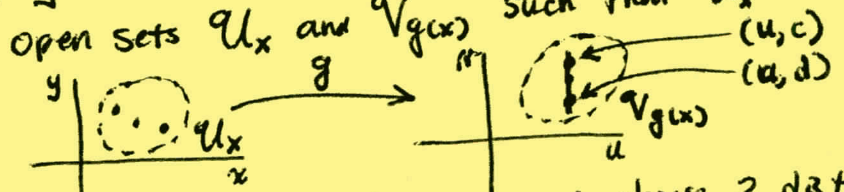
choose a pt $x \in B$ and assume $f_x(x) \neq 0$

Since f_x is Cont, by persistence of sign thm, $\exists$ an open set $\mathcal{A}$ containing this x where $f_x \neq 0$.

Define $g: \mathcal{A} \to \mathbb{R}^2$, $\mathcal{A} \subset \mathbb{R}^2$, $(x,y) \mapsto \begin{bmatrix} f(x,y) \\ y \end{bmatrix}$

Then for our x $Dg_x = \begin{bmatrix} f_x & f_y \\ 0 & 1 \end{bmatrix}$ $\det(Dg_x) = f_x(x) \neq 0$

apply Inv Fcn Thm: $\exists$ open sets $\mathcal{U}_x$ and $\mathcal{V}_{g(x)}$ such that $\mathcal{U}_x \xrightarrow{g \text{ diffeo}} \mathcal{V}_{g(x)}$



Since $\mathcal{V}_{g(x)}$ is an open set, it contains a vertical line segment. Choose 2 distinct pts on this segment (u,c) and (u,d). Then diffeo g^{-1} maps them back to distinct pts, say (p,q) and (r,s). But since they must map forward to the vertical segment, $f(p,q) = u$ and $f(r,s) = u$ $\Rightarrow\Leftarrow$ so f can't be One-to-One. $\square$

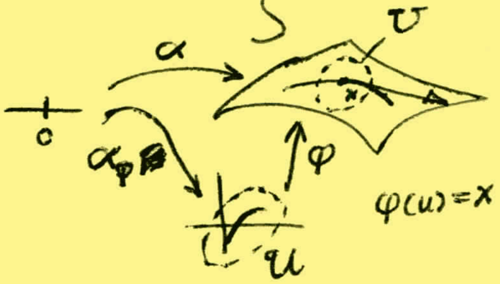
[Same idea if $f_y(x) \neq 0$]

Spivak 2.37 cont'd (b) Now generalize this for smooth maps $f : \mathbb{R}^n \to \mathbb{R}^m$, $n > m$
I attempted this in my 3/19/1992 work, but I'm not going to repeat it here.
We show $\exists$ a pt x where Df_x has rank m and then construct a fcn g which will preserve this $m \times m$ subdet when we do $\det(Dg_x)$. See my pf of Local Immersion and Submersion thms, which are coming up. $\square$

ch 1.2 #12 Show that we can also define T_xX in terms of curves on mfd X.

Let $C_x := \left\{ \text{all vectors } \vec{\alpha}'(0) \;\middle|\; \begin{array}{l} \alpha : (-\varepsilon, \varepsilon) \to X \text{ curve on mfd} \\ \alpha(0) = x \end{array} \right\}$

Show $T_xX = C_x$



$\triangleright$ show $T_xX \subseteq C_x$:

Choose $\vec{w} \in T_xX$
By def $\exists\ h \in \mathbb{R}^k \ni d\varphi_u(h) = w$
Define a curve $\gamma : I \to \mathcal{U}$, $t \mapsto \vec{u} + t\vec{h}$ then $\gamma(0) = u$, $\gamma'(0) = h$

Push γ up to a curve on X by defining $\alpha = \varphi \circ \gamma$
$\Rightarrow \alpha'(0) = D\alpha_0 = d\varphi_u(\gamma'(0)) = d\varphi_u(h) = w$ $\square$

$\triangleright$ show $T_xX \supseteq C_x$:

Let α be a curve in C_x, thus $\alpha : I \to X$, $\alpha(0) = x$, $\vec{\alpha}'(0) = w$
want to show $\exists\ v \in \mathbb{R}^k \ni d\varphi_u(v) = \vec{w}$
Define α_φ by $\alpha_\varphi = \varphi^{-1} \circ \alpha$ this is a curve down in $\mathcal{U}$.
Thus $\alpha = \varphi \circ \alpha_\varphi$
$\alpha'(0) = d\varphi_{\alpha_\varphi(0)}(\alpha_\varphi'(0)) = d\varphi_u(\alpha_\varphi'(0))$ take $v = \alpha_\varphi'(0)$ $\square$

In my write-up sheets for DoCarmo DGOCAS, I gave a discussion of comparing this formulation of the tangent space T_xX with the formulation for an abstract mfd.

$\triangleright$ a brief digression about the graph of $f : X \to Y$

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ \varphi\uparrow & & \uparrow\psi \\ \mathcal{U} & \longrightarrow & \mathcal{V} \end{array}$$

ch 1.1 #17 Define $Gr(f) := \{\text{all pairs } (x, f(x)) \in X \times Y \mid x \in X\} \subseteq X \times Y$

Define $F : X \to Gr(f) \subseteq X \times Y$, $x \mapsto (x, f(x))$. Show F is a diffeo.

(i) $F = \mathrm{Id} \times f$ so obviously it is smooth
(ii) Even if f is not one-to-one, so long as f is a fcn, F is one-to-one.
(iii) $F^{-1} : Gr(f) \to X$, $(x, f(x)) \mapsto x$ so it is just $\pi : X \times Y \to X$ restricted to $Gr(f)$, and we know proj π is smooth.

$\Rightarrow F$ is a diffeo $F(X) = Gr(f)$ $\square$

Ch 1.2 # 11 From prev problem, show $dF_x(v) = (v, df_x(v))$

Observe

$$\begin{tikzcd} & Y \\ X \arrow[ur, "f"] \arrow[r, "F"] \arrow[dr, "Id"'] & X \times Y \arrow[u, "\pi_Y"'] \arrow[d, "\pi_X"] \\ & X \end{tikzcd}$$

Then $f = \pi_Y \circ F$
$I = \pi_X \circ F$

$F(X) = Gr(f) \subseteq X \times Y$

apply chain rule (we know proj is smooth by ch 1.1 #5)

$df_x(\cdot) = d\pi_Y \circ dF_x(\cdot) \Rightarrow dF_x(v) = (?, df_x(v))$ we see this forces the 2nd component to be $df_x(v)$.

Likewise $I = d\pi_x \circ dF_x(\cdot) \Rightarrow I(v) = d\pi_x(dF_x(v)) \Rightarrow$ 1st component must be v

Put it together: $dF_x(v) = (v, df_x(v))$

(b) Show $T_{F(x)}(Gr(f)) = Gr(df_x)$ we know $Gr(df_x) = \left\{ \text{all } (v, df_x(v)) \in T_xX \times T_yY \right\}$

what is $T_{F(x)}Gr(f)$? we could derive this from charts, but instead, note: If $g: S \longrightarrow M$ is a diffeo, then $dg_s : T_sS \longrightarrow T_{g(s)}M$ is an iso.

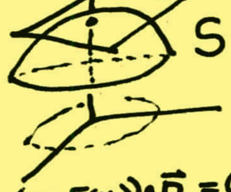
Here we have $F: X \xrightarrow{\text{diffeo}} Gr(f) \Rightarrow \boxed{dF_x(T_xX) = T_{F(x)}(Gr(f))}$

$= T_{(x,f(x))}(Gr(f))$

and in (a) we just showed $dF_x(v) = (v, df_x(v))$

$$\boxed{\text{Thus } T_{(x,f(x))}(Gr(f)) = Gr(df_x)}$$

▷ Now let's apply this to graphs of Surfaces as discussed in Marsden & Tromba VC ch 2.3, 2.5 and ch 6



$F: \mathcal{U} \longrightarrow \mathbb{R}^2 \times \mathbb{R}$ where $f: \mathcal{U} \to \mathbb{R}$ then $DF_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_x(x) & f_y(x) \end{bmatrix}$

$(x,y) \longmapsto \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$ $\quad z = f(x,y)$

Let $X_0 = \langle x_0, y_0, z_0 \rangle$. They define $T_{x_0}S$ as the set of pts $\langle x, y, z \rangle$ that satisfy $(x - F(x_0)) \cdot \vec{n} = 0$

i.e. $\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ f(x_0,y_0) \end{bmatrix} \right) \cdot \vec{n} = 0$ where normal vector to surf $\vec{n}_{x_0} = \begin{bmatrix} 1 \\ 0 \\ f_{x_0} \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ f_y \end{bmatrix} = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix}$

which works out to $z = z_0 + (x - x_0) f_x(x_0) + (y - y_0) f_y(x_0)$

This is placing the tan plane at the pt $F(x_0)$ [which G&P write as $x + T_xX$] but typically the eqs in G&P put it at the origin. So if we do that here — take $F(x_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \vec{x} \cdot \vec{n}(x_0) = 0 \Rightarrow [x\ y\ z] \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix} = 0 \Rightarrow \boxed{z = x f_x(x_0) + y f_y(x_0)}$ Eq of plane over xy plane.

▷ Now in G&P's language: $X = \mathcal{U}$, $Y = \mathbb{R}$, $T_xX = \mathbb{R}^2$

From above, $T_{(x,f(x))} Gr(f) = dF_x(T_xX) = dF_x(\mathbb{R}^2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ f_x & f_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ x f_x + y f_y \end{bmatrix}$

SAME

□