Griffel AFA
ch 2

Diff Eqs and Green's Fcns

12/2/2025
REDRAUM

p.22 $\underset{\sim}{Df}(\varphi) = -\langle\langle \underset{\sim}{f}, \varphi \rangle\rangle$ or better $\Lambda_{Df}(\varphi) = -\Lambda_f(\varphi')$

because $\Lambda_{f'}(\varphi) = \int_{-\infty}^{\infty} f'\varphi\, dx = -\int_{-\infty}^{\infty} f\varphi'\, dx$ ← int by parts, ∂ terms vanish

Def p.18 of making a distrib out of a integrable fcn f: $\Lambda_f(\varphi) := \int_{-\infty}^{\infty} f\varphi\, dx$ ← 'regular' distrib since it comes from a fcn; δ does not (Def p.20)

p.39 Linear ODEs

$$a_0 u^{(n)} + a_1 u^{(n-1)} \dots + a_n(x)\, u = f \qquad (a_0(x))$$

$$Lu = f \quad \text{but we shall go to} \quad L\underset{\sim}{u} = \underset{\sim}{f}$$

If $Lu = f$ for fcns, then $L\Lambda_u = \Lambda_f$ (Let's just consider $L = g\frac{d}{dx}$)

$$(L\Lambda_u)(\varphi) = (-1)^1 \Lambda_u\left(\frac{d}{dx}(g\varphi)\right) = -\int_{\mathbb{R}} (g\varphi)'\, u\, dx = +\int_{\mathbb{R}} \varphi\, \overset{f}{gu'}\, dx = \Lambda_f(\varphi) \checkmark$$

$$\left(g\left(\frac{d}{dx}\Lambda_u\right)\right)(\varphi) = \left(\frac{d}{dx}\Lambda_u\right)(g\varphi) \quad \text{p.19 def 1.17}$$

$$= (-1)^1 \Lambda_u\left(\frac{d}{dx}(g\varphi)\right)$$

given smooth fcn h: $h\Lambda_f(\varphi) := \Lambda_f(h\varphi) = \Lambda_{hf}(\varphi)$, $\cancel{h\int_{\mathbb{R}} f\varphi\, dx}$

A generalized soln is $\underset{\sim}{u}$ satisfying $L\underset{\sim}{u} = \underset{\sim}{f}$

A classical soln is fcn $u \ni Lu = f$

a weak soln is a fcn not C^n smooth but

If $\underset{\sim}{u}$ is a distrib soln, but $\underset{\sim}{u} \neq \Lambda_u$ for fcn u, then $\underset{\sim}{u}$ is a singular soln

2.3 Fund Solns for ODEs

Here is the heuristic

If $u = \sum a_i w_i$ and $f = \sum a_i \delta_i$ (same a_i)

then $L(\sum a_i w_i) = \sum a_i \delta_i \Rightarrow \sum a_i (Lw_i - \delta_i) = 0$

Solve $Lw_i = \delta_i$ and plug w_i in to get $\sum a_i w_i = u$

Now for our problem: $f(x) = \int_{\mathbb{R}} \overset{a_i}{f(y)}\, \delta(x-y)\, dy = \text{"}\sum_y f(y)\, \delta_y\text{"}$

Let $u(x) = \int_{\mathbb{R}} f(y)\, w(x;y)\, dy = \text{"}\sum_y f(y)\, w(x;y)\text{"}$

Then we solve $Lw(x;y) = \delta(x-y)$ and plug w back here to get u

Summarize δ fcn

$$f(x) = \int_{-\infty}^{\infty} \delta(x-y)\, f(y)\, dy \text{ "}= \sum_y f(y)\delta_y\text{"}$$

this is a point value?

$$Lw(x;y) = \delta(x-y)$$

This is a Fundamental soln — building block of soln

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②

δ-fcn only really has meaning inside an integral

Test fcns

$\mathcal{D}$ or $\mathcal{D}(\mathbb{R})$

Rudin $C_0^\infty(\mathbb{R})$

C^∞ fcns with cpt supp

(P.17) $\underset{\sim}{\delta}(\varphi) = \varphi(0)$

(P.18) Regular Distrib $\longrightarrow$ $\exists$ locally integrable fcn f

Rudin $\Lambda_f(\varphi) := \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx \quad =: \langle f, \varphi\rangle = \underset{\sim}{f}(\varphi)$

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often inner prod $\langle f, g\rangle = \int_{\mathbb{R}} f\bar{g}$ so that is why this notation exists

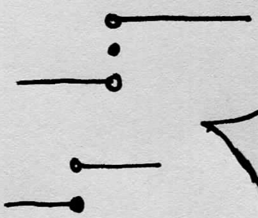
A distrib (like δ) which is not Λ_f for a fcn f is called a Singular distrib

we do write ~~$f(x) = \int_{-\infty}^{\infty} f(x)\delta(x)\,dx$~~

$\varphi(0) = \int_{\mathbb{R}} \delta(x)\varphi(x)\,dx$ — δ is not a true fcn but this formalism is its bread an butter

(P.19) Heaviside fcn $H(x) := \begin{cases} 0 & (-\infty, 0) \\ 1/2 & x = 0 \\ 1 & (0, \infty) \end{cases}$

$H_1(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$



$\Lambda_H(\varphi) = \underset{\sim}{H}(\varphi) = \int_{\mathbb{R}} \varphi(x) H(x)\,dx = \int_0^\infty \varphi(x)\, 1\, dx$

(P.21) translation COV

$\underset{\sim}{f_a}(\varphi) := \underset{\sim}{f}(\varphi \circ \tau_a)$ where $(\varphi \circ \tau_a)(x) = \varphi(x - a)$ so $x = a$ is "zero"

mult COV

$\underset{\sim}{f_{\cdot a}}(\varphi) := \underset{\sim}{f}(\varphi_{1/a})$ where $\varphi_{1/a}(x) = \frac{1}{|a|}\varphi(\frac{1}{a}x) \quad \forall\ \varphi \in \mathcal{D}$

For a reg distrib $\Lambda_{f_{\cdot a}}(\varphi) = \int_{\mathbb{R}} f(ax) \frac{1}{|a|} \varphi(\frac{x}{a})\,dx$

this would be std COV

$u = ax$

$du = a\,dx$

$\int_{x=-\infty}^{\infty} f(ax)\varphi(x)\,dx = \frac{1}{a}\int_{u=-\infty}^{\infty} f(u)\,\varphi(\frac{u}{a})\,\overbrace{a\,dx}^{du}$

example 1.24

$\delta_{(x-a)}(\varphi) = \delta_x(\varphi(x+a)) = \varphi(a)$

$\delta_{ax}(\varphi) = \delta_x\left(\frac{1}{|a|}\varphi(x/a)\right) = \frac{1}{|a|}\varphi(0)$

ie "$\delta_{(ax)} = \frac{1}{|a|}\delta_x$"

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$$\Lambda_f(\varphi) := \int_{\mathbb{R}} f\varphi\, dx \qquad \Lambda_{f'}(\varphi) = \int_{-\infty}^{\infty} f'\varphi\, dx = -\int_{-\infty}^{\infty} f\varphi'\, dx$$

int by parts
∂ terms vanish

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example 1.27

$x \longmapsto |x|$ is locally integ fcn

$$\left(\frac{d}{dx}\Lambda_{|x|}\right)(\varphi) = -\Lambda_{|x|}(\varphi') = -\int_{-\infty}^{\infty} |x|\,\varphi'(x)\,dx$$

$$= \int_{-\infty}^{0} x\,\varphi'(x)\,dx - \int_{0}^{\infty} x\,\varphi'(x)\,dx \quad \text{elim abs val}$$

$$= -\int_{-\infty}^{0} \varphi(x)\,dx + \int_{0}^{\infty} \varphi(x)\,dx \quad \text{int by parts again here } \varphi \text{ vanishes at } \infty$$

$$\mathrm{Sgn}(x) := \begin{cases} -1 & x \text{ neg} \\ +1 & x \text{ pos} \end{cases}$$

$$\Lambda_{|x|'} = \Lambda_{\mathrm{sgn}}$$

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ex 1.28

Heaviside H

$$\left(\frac{d}{dx}\Lambda_H\right)(\varphi) = -\Lambda_H(\varphi') = -\int_{\mathbb{R}} H(x)\,\varphi'(x)\,dx = -\int_{0}^{\infty} \varphi'(x)\,dx$$

$$= -\left[\cancel{\varphi(\infty)}^{\,0} - \varphi(0)\right] = \varphi(0) = \delta_0(\varphi)$$

$$\Rightarrow \boxed{\frac{d}{dx}\Lambda_H = \delta} \quad \text{or} \quad \underset{\sim}{H}'(x) = \underset{\sim}{\delta}(x)$$

$\triangleright$ $\left(\frac{d}{dx}(h\underset{\sim}{f})\right)(\varphi) = -(h\underset{\sim}{f})(\varphi') = -\underset{\sim}{f}(h\varphi')$ ...... $h\left(\frac{d}{dx}\underset{\sim}{f}\right)(\varphi) + h'\underset{\sim}{f}(\varphi)$

Prod Rule

$$= \left(h\frac{d}{dx}\underset{\sim}{f} + h'\underset{\sim}{f}\right)(\varphi)$$

~~Stakgold~~

Stakgold, I.

BVPs of Math Phys

1967–68

Green's Fcns and BVPs

1979

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$-\nabla^2 u = f$ converted to distribs $-\underset{\sim}{\nabla}^2_x \underset{\sim}{w}(x,y) = \underset{\sim}{\delta}_{(x-y)}$ (3.32)

Big Thm & p.72

soln $w(x;y) = \dfrac{1}{4\pi \|x-y\|}$ ES potential at x of pt charge located at y

$u(x) = \dfrac{1}{4\pi}\displaystyle\int \frac{1}{\|x-y\|} f(y)\, d^3y$ ES potential at x for charge region with density fcn $f(y)$.

Now we must rigorously prove $\boxed{\text{Thm 3.31 } \Lambda_{\frac{1}{4\pi\|x-y\|}} \text{ is fund soln for operator } L = -\nabla^2 \text{ in } \mathbb{R}^3}$

Step 1 Fix $y \in \mathbb{R}^3$ $\quad w_y := \dfrac{1}{4\pi\|x-y\|}$

we must show $(-\nabla^2 \Lambda_{w_y})(\varphi) = \delta_y(\varphi) \quad \forall\ \varphi \in \mathcal{D}(\mathbb{R}^3)$

$= -(-1)^2 \Lambda_{w_y}(\nabla^2\varphi)$

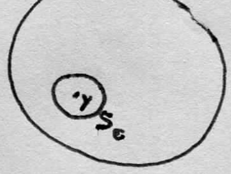
want to evaluate this

$= \dfrac{-1}{4\pi}\displaystyle\int_{\mathbb{R}^3} \nabla^2\varphi \frac{1}{\|x-y\|}\, d^3x$ $\quad \varphi$ has cpt supp, $\text{sup}(\varphi) \subset \overbrace{B(0,R)}^{G}$ for R large enough

Take R also big enough that $y \in G$

$= \lim\limits_{\varepsilon \to 0} \dfrac{-1}{4\pi}\displaystyle\int_{G_\varepsilon} \nabla^2\varphi \frac{1}{\|x-y\|}\, d^3x$ — Call this $R^{-1/2}$

Step 2 want to apply Green's Id $\quad G_\varepsilon := B(0,R) - B(y,\varepsilon)$



By computing partials note $\nabla^2(R^{-1/2}) = 0$

$-\dfrac{1}{4\pi}\displaystyle\int_{G_\varepsilon}\Big(\nabla^2\varphi \overset{1/r}{R^{-1/2}} + \underbrace{\varphi \nabla^2 \overset{1/r}{(R^{-1/2})}}_{\text{adding } 0}\Big) dV \overset{\text{Green}}{=\!=} \dfrac{-1}{4\pi}\displaystyle\int_{\partial G_\varepsilon}\Big(R^{-1/2}\nabla\varphi - \varphi\nabla(R^{-1/2})\Big)\cdot n\, dS$

so we have

Step 3 Reduce ∂ to only S_ε

$(-\nabla^2\Lambda_{w_y})(\varphi) = \lim\limits_{\varepsilon \to 0} \dfrac{-1}{4\pi}\displaystyle\int_{G_\varepsilon} (R^{-1/2}\nabla\varphi - \varphi\nabla R^{-1/2})\cdot \hat{n}\, dS$ but $\partial G_\varepsilon = \partial G \uplus \partial S_\varepsilon$ and on ∂G $\varphi \equiv 0$, $\nabla\varphi \equiv 0$ cpt supp

$= \lim\limits_{\varepsilon \to 0} \dfrac{-1}{4\pi}\displaystyle\int_{S_\varepsilon} (R^{-1/2}\nabla\varphi - \varphi\nabla(R^{-1/2}))\cdot n\, dS$

$\downarrow$

$\dfrac{-1}{4\pi}\displaystyle\int_{G_\varepsilon}\Big(\frac{1}{r}\nabla^2\varphi + \varphi\nabla^2(1/r)\Big)dV = \dfrac{-1}{4\pi}\displaystyle\int_{\partial G_\varepsilon}\Big(\frac{1}{r}\nabla\varphi - \varphi\nabla(1/r)\Big)\cdot n\, dS$

$\nabla(1/r) = \dfrac{-1}{r^3}\vec{r} = \dfrac{-1}{\varepsilon^2}\hat{v}$ where Plug in $r = \varepsilon$ on ∂S_ε

so $x = y + \varepsilon v$ and $\hat{v} = -n_x$ $\quad r =$

Outward pointing normal $\quad \hat{v} := \dfrac{\vec{x} - \vec{y}}{\|x-y\|} = \dfrac{1}{\varepsilon}(x-y)$

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Step 4 Convert integration to spherical coords w/ y as origin

$$\lim \frac{-1}{4\pi} \int_{S_\varepsilon} \left(\left(R^{-1/2} \nabla\varphi - \varphi \nabla (R^{-1/2}) \right) \right) \cdot n \, dS = \frac{-1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \left(\frac{1}{\varepsilon} \nabla\varphi(x) + \varphi(x) \left(\frac{+1}{\varepsilon^2} \right) \hat{v} \right) \cdot -\hat{v} \, \varepsilon^2 \sin\beta \, d\alpha \, d\beta$$

no $d\varepsilon$
ε is fixed radius

$$= \frac{+1}{4\pi} \iint \left(+ \varepsilon \nabla\varphi \cdot \hat{v} + \underbrace{\|v\|^2}_{1} \varphi \right) \sin\beta \, d\alpha \, d\beta$$

now recall $x = y + \varepsilon v$ and $\varphi : \mathbb{R}^3 \to \mathbb{R}$ so we can apply MVT

$$\varphi(y+\varepsilon v) - \varphi(y) = D\varphi_\lambda(\varepsilon v) \text{ for some } \lambda \text{ on line seg } y \text{ to } y+\varepsilon v \quad [y, y+\varepsilon v]$$

$$= \varepsilon \nabla\varphi_{(\lambda)} \cdot v$$

$$\text{Integrand} = \varepsilon \nabla\varphi(y+\varepsilon v) \cdot v + \varepsilon \nabla\varphi(\lambda) \cdot v + \varphi(y)$$

$$\Rightarrow \underbrace{\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \varphi(y) \sin\beta \, d\alpha \, d\beta}_{\varphi(y)} + \underbrace{\frac{1}{4\pi} \iint \varepsilon \left(\nabla\varphi(y+\varepsilon v) + \nabla\varphi(\lambda) \right) \cdot \hat{v} \sin\beta \, d\alpha \, d\beta}_{\to 0 \text{ as } \varepsilon \searrow 0}$$

$$\Rightarrow \left(-\nabla^2 \Lambda_w \right)(\varphi) = \delta_y(\varphi)$$

$\boxed{\text{QED}}$

2D

$\int$