Fowles AM ch 6
p.135 and prob 6.2

9/25/2007

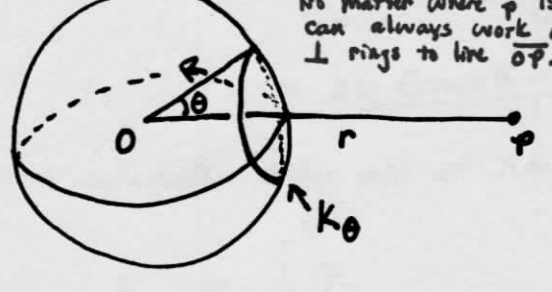
Gravitation Attraction of Spherical Shell by ① Forces, ② Potential and ③ Gauss' Law

method attributed to Thompson & Tait by F. Moulton.

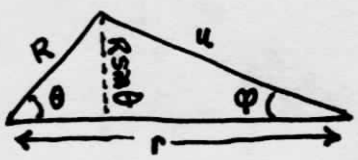
Results: (1) Outside sphere the field is that of a pt. mass
(2) Inside sphere the field is 0 at all pts.

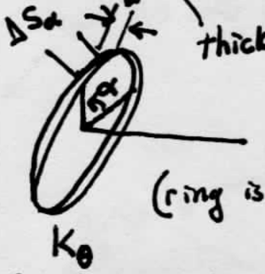
▷ Derive results directly from Forces (possible since we have symmetry):

First use of Symm (S^2-symm): No matter where p is, we can always work with ⊥ rings to line $\overline{OP}$.



By rotational symm, we can always slice the sphere into rings K_θ which are ⊥ to the radial lines from the center O to pt. p (particle of mass m).





thickness $R\Delta\theta =: \Delta S_\theta$

$\Delta S_\alpha = (R\sin\theta)\Delta\alpha$

(ring is drawn too tilted)

K_θ

What is the mass of an elt of the ring K_θ? Let ρ be the surf density.

$$\Delta M_{(\alpha,\theta)} = \rho(\Delta S_\alpha)(\Delta S_\theta) = \rho(R\sin\theta\,\Delta\alpha)(R\Delta\theta)$$

Newton's Law of Grav is
$$F = G\frac{Mm}{r^2}$$

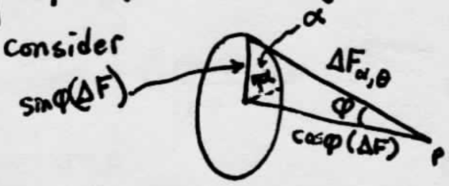
so the force between the ring elt and p is: $\Delta F_{\alpha,\theta} = \dfrac{G(\Delta M_{\alpha,\theta})m}{u^2}$

▷ claim: By rotational symm about the line $\overline{OP}$, the force due to all of K_θ is directed along $\overline{OP}$ (thus the ring attracts p toward its center):

So we won't need to write the vector symbol any more because we know which way it is pointing.

2nd use of Symm (S^1-symm): we can reduce the force of ring to a vector along $\overline{OP}$.

To see this, consider $\Delta F = \Delta F^{\parallel} + \Delta F^{\perp}$



φ is related to θ, see above diagram.

We can phrase the symmetry arg in 2 ways:

① The ring is param'ed by $\alpha \in (-\pi, \pi]$. For each pos α_0, the component of ΔF $\underbrace{\text{that is } \perp \text{ to } \overline{OP}}_{(\sin\varphi)\,\Delta F_{\alpha_0,\theta}}$, is countered by the $\underbrace{\perp \text{ component from } -\alpha_0}_{(\sin\varphi)\,\Delta F_{-\alpha_0,\theta}}$.

'Jeff symm'

② The system $\{K_\theta$ & $p\}$ is invariant under rotations about $\overline{OP}$. Thus the net force $\vec{F}_{K_\theta}$ must be invariant under rotations too, so it can only be along $\overline{OP}$.

Thus $F_{k_\theta} = \int_{\alpha=-\pi}^{\pi} \cos\varphi \,(\Delta F_{\alpha,\theta})$

$$= \int \cos\varphi \frac{G\,\Delta M_{\alpha,\theta}\, m}{u^2}$$

$$= \int_{-\pi}^{\pi} \frac{\cos\varphi\, G m \rho R \sin\theta\, R\Delta\theta}{u^2}\, d\alpha \longleftarrow \text{we turned } \Delta\alpha \text{ into } d\alpha$$

$$= \frac{2\pi\, G m \rho R^2 \sin\theta \cos\varphi\, \Delta\theta}{u^2}$$

Now integrate over all of the K_θ rings i.e. vary θ: remember u and φ depend on θ

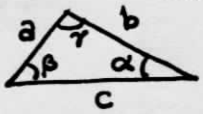
$$F = \int_{\theta=0}^{\pi} F_{k_\theta}$$

turn $\Delta\theta$ into $d\theta$

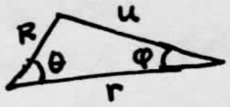
$$= 2\pi G\rho m \int_0^{\pi} \frac{R^2 \sin\theta \cos\varphi\, d\theta}{u^2}$$

Now we need some trickery to eval this:

Law of Cos



$a^2 = b^2 + c^2 - 2bc\cos\alpha$



It turns out we want

(*) $R^2 = r^2 + u^2 - 2ru\cos\varphi$

(**) $u^2 = R^2 + r^2 - 2Rr\cos\theta$

differentiate (**): $2u\,du = -2Rr(-\sin\theta)\,d\theta$

$\Rightarrow \frac{u}{r}du = R\sin\theta\, d\theta$

$$= 2\pi G\rho m \int_{\theta=0}^{\pi} \frac{R\cos\varphi}{u^2}\frac{u}{r}\,du$$

Now change limits of integration:

(**) $u = \pm\sqrt{R^2 + r^2 - 2Rr\cos\theta}$ distance

$\theta = 0 \Rightarrow u = \sqrt{R^2+r^2-Rr} = \sqrt{(R-r)^2} \Rightarrow u = \begin{cases} r-R & r>R \\ R-r & r<R \end{cases}$ if p outside / p inside

$\theta = \pi \Rightarrow u = \sqrt{R^2+r^2+Rr} \Longrightarrow u = R+r$

$$= 2\pi G\rho m \int_{u=A}^{R+r} \frac{R\cos\varphi}{ur}\,du$$

$A = r-R$ outside, $R-r$ inside

using (*) $\frac{-R^2+r^2+u^2}{2ru} = \cos\varphi$

$\frac{R}{ur}[\cos\varphi] = \frac{R}{2ur}\left[\frac{r}{u} + \frac{u}{r} - \frac{R^2}{ur}\right]$

$$= 2\pi G\rho m \int_{u=A}^{R+r} \frac{R}{2}\left[\frac{1}{r^2} + \frac{1}{u^2}\left(1 - \frac{R^2}{r^2}\right)\right]du$$

$$= \cancel{2}\pi G\rho m \frac{R}{\cancel{2}}\left[\frac{1}{r^2}u - \left(1-\frac{R^2}{r^2}\right)\frac{1}{u}\right]_A^{R+r}$$

let $P := R+r$, $M := R-r$ then p inside $\Rightarrow A = M$, outside $\Rightarrow A = -M$

$$= \frac{\pi G\rho m R}{r^2}\left[P - A - MP\left(\frac{1}{A} - \frac{1}{P}\right)\right]$$

<u>outside $A = -M$:</u>

$\frac{\pi G\rho m R}{r^2}\underbrace{\left[P + M + MP\left(\frac{1}{M}+\frac{1}{P}\right)\right]}_{= 2(P+M) = 4R}$

area of sphere! $4\pi R^2$

$= \frac{G(\rho 4\pi R^2)m}{r^2} = \boxed{\frac{GMm}{r^2}}$ just like for a pt mass!

<u>Inside $A = M$</u>

$\left[P - M - MP\left(\frac{1}{M} - \frac{1}{P}\right)\right]$

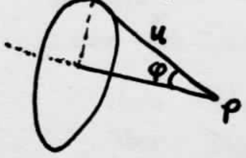
$P - M - (P - M)$

$= 0$ for p <u>anywhere</u> inside. $\square$

Cont'd →

Now show it from potential fcn considerations:

For a pt. on the ring K_θ, we know $V = \frac{G(\Delta M_{\alpha,\theta})m}{r}$ and $\vec{F} = -\nabla V$



Thus we can repeat our procedure (the only difference is now we have r^{-1} vs. r^{-2})

No concern for symm now!

$$\Delta M_{\alpha,\theta} = \rho\,(R\sin\theta\,\Delta\alpha)(R\Delta\theta)$$

then

$$V_{K_\theta} = \int_{\alpha=-\pi}^{\pi} \frac{G(\Delta M_{\alpha,\theta})\,m}{u^{\cancel{2}}}\,d\alpha$$

$$= \frac{G\rho R^2 \sin\theta\,\Delta\theta\,2\pi}{u}$$

$$V = 2\pi G\rho m \int_{\theta=0}^{\pi} \frac{R^2 \sin\theta\,d\theta}{u}$$

Subs like before

$\frac{u}{r}du = R\sin\theta\,d\theta$

limits of integration are the same too:

$\theta = 0 \Rightarrow u = A$

$\theta = \pi \Rightarrow u = R+r$

$$= m\,2\pi RG\rho \int_{u=A}^{R+r} \frac{1}{\cancel{u}}\frac{\cancel{u}}{r}\,du$$

$$= \frac{m\,2\pi RG\rho}{r}\left[R+r-A\right]$$

Outside

Just like before $A = -M$
$= r-R$

$R+r-(r-R)$
$= 2R$

$$\Rightarrow \frac{2\pi\,m\,GR\rho(2R)}{r}$$

$$= \frac{Gm(\rho 4\pi R^2)}{r}$$

$V(r) = \frac{GmM}{r}$ for any pt outside.

same potential as for a pt. mass!

Inside

$A = M = R-r$

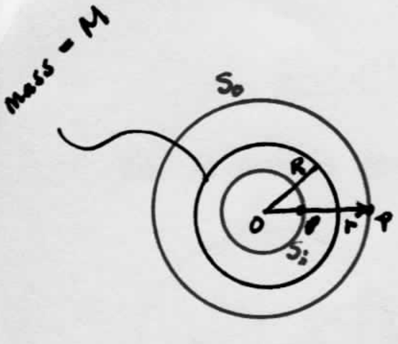
$$\Rightarrow \frac{m\,2\pi RG\rho(2\cancel{r})}{\cancel{r}}$$

$= G\rho(4\pi R)$ const

$V(r) = $ const for any pt inside.

$\Rightarrow F = -\nabla V = 0$ □

Now derive these results very easily from Gauss' Law for Gravity



From Marsden & Tromba VC ch 7.4 #16 we can use Gauss' Law for Gravity for a sphere (due to the symm)

Just identify mass M with pos charge Q.
Then Gauss becomes $\int_S \vec{F}_G \cdot \hat{n}\, dA = M_{inside}$

From S^2-symm, we know $\vec{F} = F(r)\hat{e}_r$; only dependence on r, homogeneous wrt θ, φ. spherical co-ord angles.

▷ Outside
Now consider a larger concentric sphere S_0 that touches p. $S_0 = S^2_{(0,r)}$

Gauss: $M_{inside} = \int_{S_0} F(r)\hat{e}_r \cdot \hat{e}_r\, dA = F(r)\int_0^{2\pi}\int_0^{\pi} 1 \cdot r^2 \sin\varphi\, d\theta\, d\varphi = F(r)\, r^2 4\pi$

$$\Rightarrow \boxed{\frac{1}{4\pi}\frac{M}{r^2} = F(r)}$$

Now consider another mass, test mass m at position vector $\vec{r} = r\hat{e}_r$. By linearity of $\vec{F}_G$

Force on m due to sphere. $\boxed{F = \frac{1}{4\pi}\frac{mM}{r^2}}$

M&T VC ch 7.3 #16

▷ Inside
Same idea, now $r<R$
No mass inside

$$0 = \int_{S_i} F(r)\hat{e}_r \cdot \hat{e}_r\, dA \Rightarrow 0 = F(r)\, r^2 4\pi$$

$$\Rightarrow \boxed{F(r) = 0 \quad \text{for any } r<R}$$

▷ Does Gauss' Law show that the grav field inside an arb. empty cavity is 0, like it does for electrostatic field in a conductor?

No!
a body of mass does not correspond to a conductor, it would correspond to a solid body of pos charge. In electrostatics, the key idea is that the E field must be 0 in a conductor since no charge is flowing, and this makes $E=0$ in any cavity.