Remark: If I can get work out of a System, the Potential Energy is the work that (presumably) had to be put into the system to get it into that configuration.

See my Fowles AM ch 7 write up May 2009

1/20/2001

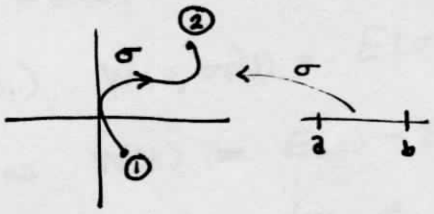
The Generalized Work - (kinetic) Energy Thm And Conservation of Energy For A "Point Mass"

For systems of particles, see Benson UP p. 202

Consider a path σ in the configuration space of a system with a particle of mass m and a conservative force vf F [$F = -\nabla U$ for some $U: \mathbb{R}^3 \to \mathbb{R}$] acting on the particle. Let us assume there is also an external force F_e

We want to establish

$$\boxed{P_1 + K_1 + W_{ext} = P_2 + K_2}$$



▷ consider the following important calculation:

Route A:

$$-\int_\sigma \nabla U \cdot ds = -\int_a^b \nabla U(\sigma(t)) \cdot \sigma'(t)\, dt = U(a) - U(b) = -\Delta(PE)$$

$$\text{Work } W_F = \int_\sigma F \cdot ds$$

Route B:

$$\int_a^b F(\sigma(t)) \cdot \sigma'(t)\, dt$$

$F \cdot v = \frac{d}{dt}(\frac{1}{2} m v \cdot v)$ Fowles AM p. 86-87

$$= \frac{m}{2}\int_a^b \frac{d}{dt}\left(\|\varphi'(t)\|_2^2\right) dt = \frac{1}{2} m \|\dot\varphi(b)\|^2 - \frac{1}{2} m \|\dot\varphi(a)\|^2 = \Delta(KE)$$

There is a chain of equalities connecting the results of Route A and Route B, thus we have established

$$\boxed{\begin{aligned} U(a) + \tfrac{1}{2} m V^2(a) &= U(b) + \tfrac{1}{2} m V^2(b) \\ P_1 + K_1 &= P_2 + K_2 \end{aligned}}$$

Total Mechanical Energy is conserved for work done by the conservative force $F = -\nabla U$

▷ Route B holds for any force F, not just ones that came from a potential. In particular, if we let F_R denote the resultant of all forces acting on the particle, we get the Work - (kinetic) Energy Thm (H&R p. 99)

$$W_{F_R} = \int_\sigma F_R \cdot ds = \frac{m}{2}\left(V^2(b) - V^2(a)\right) = \Delta(KE)$$

Thus if there is no change in KE, there is no work done by the resultant force H&R p. 95, 92

▷ For our system then, we would have $F_R = F_e + F_U$ [F_U is just an abbreviation signifying $F = -\nabla U$]

"F_e" is all the forces on particle, that do NOT come from the potential U.

Thus

$$\int_\sigma (F_e + F_U) \cdot ds = K_2 - K_1$$

$$\Rightarrow \int_\sigma F_e \cdot ds = -\int_\sigma F_U \cdot ds + K_2 - K_1$$

$$W_{ext} = P_2 - P_1 + K_2 - K_1 \Longrightarrow P_1 + K_1 + W_{ext} = P_2 + K_2 \qquad \square$$

Remarks:

(1) A classic example of this Thm is Bernoulli's Thm in H&R or Benson. Masses & springs Strang ITAM p. 42-43

(2) More generally we would have $P_1 + K_1 + W_e = P_2 + K_2 + \text{Heat} + \text{other energy}$

(3) Conservation of Energy is the first law of Thermo: $\Delta E = Q - W$

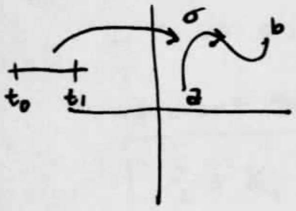
(4) Conservation of Energy is restricting the motion (in the Phase space) to a co-dim 1 Submfd.

cont'd →

Work-Energy Thm cont'd

We can also establish that power $P = \frac{dE}{dt}$

For the work done on the system by the external force along an arb path σ



$$W_e(a,b) = (P(b) - P(a)) + (K(b) - K(a))$$

$$[P(b)+K(b)] - [P(a)+K(a)]$$

$$= E(b) - E(a)$$

Then for any $t \in (t_0, t_1)$, $W(a, \sigma(t)) = E(\sigma(t)) - E(\sigma(t_0))$

which we can rewrite as $W(t) = E(t) - E(t_0)$

Then the rate at which work is being done at pt $\sigma(t)$ and time t

is $P = W'(t) = E'(t)$