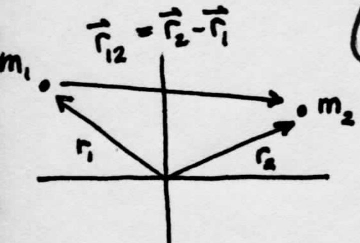
Fowles AM
ch 6

Central Force Motion

This is a special case where the attraction is about a mass fixed at the origin. The standard 2 body problem falls into this category by using the 'reduced mass'.

There is much more material for gravitation between real bodies in Roy OM and Marion CDOPAS



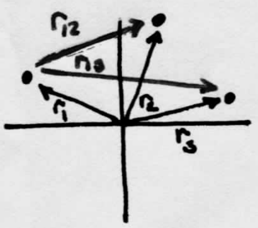
Gravity is straight line attraction (Coulomb's Law is quite similiar, but there, like charges repel)

Force on m_1 due to m_2 $\boxed{\vec{F}_{12} = \frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12}}$ where $\vec{r}_{12}$ points from m_1 to m_2

$\vec{F}_{21} = -\vec{F}_{12}$ by Newton III

Gravitational Const $G \doteq 6.672 \times 10^{-11} \frac{Nm^2}{kg^2}$

What about multiple bodies?

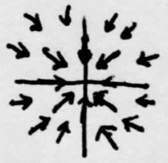


Sum all forces on particle 1: $\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13}$

$= \frac{Gm_1m_2}{r_{12}^2}\hat{r}_{12} + \frac{Gm_1m_3}{r_{13}^2}\hat{r}_{13}$

So for N particles, forces on particle i: $\boxed{\vec{F}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{Gm_im_j}{r_{ij}^2}\hat{r}_{ij}}$

▷ What we want to discuss is a fixed mass M at the origin and the gravitational field it produces that would be experienced by a mass m in space $\mathbb{R}^3 - \{0\}$



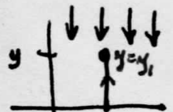
Force of attraction toward the origin $\vec{F} = -\frac{GMm}{r^2}\hat{e}_r$

▷ I'm going out-of-seq and giving discussion of Fowles ch 6.3 (Potential Energy in Grav Field) before his ch 6.2 (Grav force between unif sphere and a Particle)

WARM-UP: Unif grav field at surf of the Earth
and we assume PE is pos for a mass lifted up

A priori: Gravity points down

If we drop a mass, pos PE is converted to pos KE



The particle experiences a downward force $\vec{F}(y) = -mg\,\hat{e}_y$

Potential $V(y) = +mgy$ (ref is $y=0$ ground level)

Path $\sigma(y) = y$ $0 \le y \le y_1$
$F(y) = -mg\,\hat{e}_y$
$\int_\sigma \vec{F}\cdot d\vec{s} = \int_0^{y_1} F(\sigma(y))\,\sigma'(y)\,dy$
$= \int_0^{y_1} -mg\,\hat{e}_y \cdot 1\hat{e}_y\,dy$
$= -mgy_1$

So if we want to get $V(y) = +mgy$, we need to put a neg sign on F: $\int_0^{y_1} (-\vec{F})\cdot d\vec{s}$

We need to put a minus sign in the grad of V to recover $\vec{F}$:

$$\vec{F}(y) = -\nabla V(y) = -\begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \\ 0 \end{bmatrix} = -mg\,\hat{e}_y$$

Force should be in the direction of decreasing potential, but ∇V gives the incr dir, so we put the minus.

Certain neg signs are causing me confusion, and I want to straighten all that out. Fowles (and other books) want to talk about the work done <u>against</u> the field F moving the particle and writes $dW = -\vec{F}\cdot d\vec{r}$ This makes no logical sense to me.

But we must have ① $\vec{F} = -\dfrac{GMm}{r^2}\hat{e}_r$ ② $F = -\nabla V$ and ③ $V(r) = -\dfrac{GMm}{r}$ standard conventions in all books

▷ Let's ignore all physics conventions for the moment and just show how the math works out [Following Marsden & Tromba ch 7.3 example]

$\vec{F}(x) = \dfrac{-GMm}{\|x\|^3}\vec{x} = \dfrac{-GMm}{r^2}\hat{e}_r$ (note $\vec{r} = \vec{x}$)

Show ⓐ $\nabla\times F = 0$ in $\mathbb{R}^3 - \{0\}$ (so a potential fcn (vf generating fcn) exists there) and ⓑ Find $f \ni F = \nabla f$

ⓐ $\nabla\times F = \nabla\times\left(\dfrac{GMm}{r^3}\vec{r}\right) = GMm\left(\nabla\left(\dfrac{1}{r^3}\right)\times\vec{r} + \dfrac{1}{r^3}(\nabla\times\vec{r})\right)$

$= GMm\left(\dfrac{-3}{r^5}\underbrace{\cancel{\vec{r}\times\vec{r}}}_{0} + \dfrac{1}{r^3}\underbrace{\cancel{\nabla\times\vec{r}}}_{0 \text{ expand it out}}\right)$

$= 0$

by M&T ch 3.5 identity $\nabla(r^n) = n\,r^{n-2}\vec{r}$

Here $\nabla(r^{-3}) = -3r^{-5}\vec{r}$

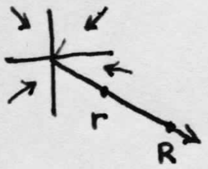
(The identity also solves ⓑ but let's do it by integration)

ⓑ Let's integrate to find the potential fcn f so $F = \nabla f$:

$\vec{F}(x) = -\dfrac{k}{\|x\|^3}\vec{x}$ $k := GMm$ This is a radial field, spherically sym.

Let's take our path to be a radial line. Fix unit vector $\hat{a}$. The line is all $x = r\hat{a}$ where r is a Real number varying in an interval. Ref pt: $R\hat{a}$

The path from $R\hat{a}$ to $r\hat{a}$ is then $\sigma(t) = (1-t)R\hat{a} + t\underbrace{r\hat{a}}_{x}$ for $t\in[0,1]$



Define $f_R(x) := \displaystyle\int_\sigma \vec{F}\cdot d\vec{s} = \int_0^1 \vec{F}(\sigma)\cdot\vec{\sigma}'\,dt = \int_0^1 \frac{-k}{\|\sigma\|^3}\vec{\sigma}\cdot\vec{\sigma}'\,dt$

$\|\sigma\| = |(1-t)R + tR|\,\|\hat{a}\| = (1-t)R + tr$ for $R, r > 0$

$\sigma' = (r-R)\hat{a}$

$= \displaystyle\int_0^1 \frac{-k}{[tr + (1-t)R]^{\cancel{3}\,2}}\cancel{(tr + (1-t)R)}\,\underbrace{\hat{a}\cdot\hat{a}}_{1}(r-R)\,dt$

$= \displaystyle\int_{t=0}^1 \frac{-k}{[tr + (1-t)R]^2}(r-R)\,dt$ COV $u := tr + (1-t)R$, $du = (r-R)\,dt$

$= -k\displaystyle\int_{u=R}^r \frac{1}{u^2}\,du = -k\left[\frac{-1}{u}\right]_R^r = k\left[\frac{1}{r} - \frac{1}{R}\right]$

Here we can't take our reference pt R as 0, or this blows up. Instead take $R = \infty$ We bring the mass m particle in from $R = \infty$

$f_\infty(x) = k\left(\dfrac{1}{r}\right) = \dfrac{GMm}{r}$

$F = \nabla f = \nabla\left(\dfrac{GMm}{r}\right) = GMm\,\nabla\left(\dfrac{1}{r}\right) = \dfrac{-GMm}{r^3}\vec{r}$ and we recover the correct $\vec{F}$!

NOTE: $\nabla\Big(\big[\underbrace{x^2+y^2+z^2}_{Q}\big]^{-1/2}\Big)$ would have, e.g. $\dfrac{\partial}{\partial x}Q^{-1/2} = -\dfrac{1}{2}Q^{-3/2}\,2x = \dfrac{1}{Q^{3/2}}x$ etc...

How do we get Work into this discussion?

$W = \Delta PE$ when there is no KE (here the particle is at rest at its initial and final positions)

We had $f_R(x) = \int_{\sigma} \vec{F}\cdot d\vec{s} = \int_{R\hat{a} \to \infty}^{r\hat{a} \to r} \vec{F}\cdot d\vec{s}$ just write end pts, since indep of path

By spherical sym, only the radial dists matter $\hat{a}$ is immaterial.

But since potential is only defined up to an additive const (which disappears when we take grad) we can write

$$f_R(x) := \int_{Ra}^{ra} F\cdot ds + C \quad \text{and let } C = f_R(R\hat{a})$$

Work $W = f_R(x) - f_R(Ra) = \int_{Ra}^{ra} \vec{F}\cdot d\vec{s}$

$f(\infty) = 0$

The way to keep all this clear is to start with the actual $\vec{F}$ felt by the particle, and integrate over the actual path i.e. ∞ to r then make changes

▷ Now introduce the physics conventions:

we require $\vec{F} \overset{!}{=} -\vec{\nabla} V$ but we already have $F = \nabla f$ for $f = \frac{GMm}{r}$

Define $V := -f = -\frac{GMm}{r}$

then $F = -\nabla V = -\nabla(-f) = \nabla f$ so it works.

To get $W = \int_{\sigma} -\vec{F}\cdot d\vec{s}$ we know

$$\int_{\infty}^{r} F\cdot ds = +\frac{GMm}{r} \quad \text{for real } \vec{F} = -\frac{GMm}{r^2}\hat{r}$$

$$= -V(r)$$

$$= -V(r) - -\underbrace{V(\infty)}_{0}$$

$$\Rightarrow \int_{\infty}^{r} (-F)\cdot d\vec{s} = V(r) - V(\infty) = \Delta PE = W$$

In other words, we are taking work W as the change in potential (V(final) − V(initial)) for this V

Fowles AM ch 6.2
p.135 and prob 6.2

9/25/2007

①

Gravitation Attraction of Spherical Shell

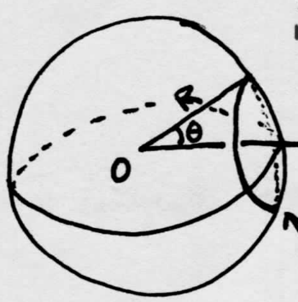
by ① Forces, ② Potential and ③ Gauss' Law

method attributed to Thompson & Tait by F. Moulton.

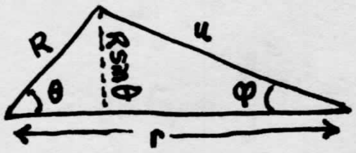
Results: (1) Outside sphere the field is that of a pt. mass
(2) Inside sphere the field is 0 at all pts.

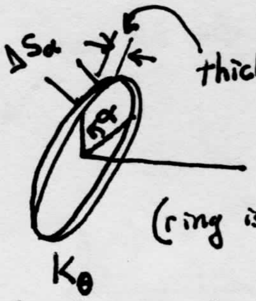
▷ Derive results directly from Forces (possible since we have symmetry):

First use of symm (S^2-symm): No matter where p is, we can always work with ⊥ rings to line $\overline{OP}$.



By rotational symm, we can always slice the sphere into rings K_θ which are ⊥ to the radial lines from the center O to pt. p (particle of mass m).





thickness $R\Delta\theta =: \Delta S_\theta$

$\Delta S_\alpha = (R\sin\theta)\Delta\alpha$

(ring is drawn too tilted)

what is the mass of an elt of the ring K_θ? Let ρ be the surf density.

$$\Delta M_{(\alpha,\theta)} = \rho(\Delta S_\alpha)(\Delta S_\theta)$$
$$= \rho(R\sin\theta\,\Delta\alpha)(R\Delta\theta)$$

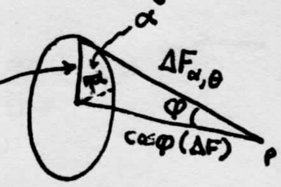
Newton's Law of Grav is
$$F = G\frac{Mm}{r^2}$$

so the force between the ring elt and p is: $\Delta F_{\alpha,\theta} = \dfrac{G(\Delta M_{\alpha,\theta})m}{u^2}$

▷ claim: By rotational symm about the line $\overline{OP}$, the force due to all of K_θ is directed along $\overline{OP}$ (thus the ring attracts p toward its center): so we won't need to write the vector symbol any more because we know which way it is pointing.

2nd use of symm (S^1-symm): we can reduce the force of ring to a vector along $\overline{OP}$.

To see this, consider $\Delta F = \Delta F^{\parallel} + \Delta F^{\perp}$



φ is related to θ, see above diagram.

we can phrase the symmetry arg in 2 ways:

① The ring is param'ed by $\alpha \in (-\pi, \pi]$. For each pos α_0, the component of ΔF that is ⊥ to $\overline{OP}$, $\underbrace{}_{(\sin\varphi)\Delta F_{\alpha_0,\theta}}$ is countered by the ⊥ component from $-\alpha_0$. $\underbrace{}_{(\sin\varphi)\Delta F_{-\alpha_0,\theta}}$

'Jeff symm'

② The system {K_θ & p} is invariant under rotations about $\overline{OP}$. Thus the net force $\vec{F}_{K_\theta}$ must be invariant under rotations too, so it can only be along $\overline{OP}$.

Thus $F_{K_\theta} = \int_{\alpha=-\pi}^{\pi} \cos\varphi \, (\Delta F_{\alpha,\theta})$

$$= \int \cos\varphi \, \frac{G \, \Delta M_{\alpha,\theta} \, m}{u^2}$$

$$= \int_{-\pi}^{\pi} \frac{\cos\varphi \, G m \rho R \sin\theta \, R \Delta\theta}{u^2} \, d\alpha$$ ⟵ we turned $\Delta\alpha$ into $d\alpha$

$$= \frac{2\pi \, G m \rho R^2 \sin\theta \cos\varphi \, \Delta\theta}{u^2}$$

Now integrate over all of the K_θ rings i.e. vary θ: remember u and φ depend on θ

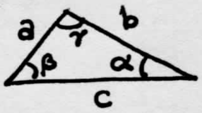
$$F = \int_{\theta=0}^{\pi} F_{K_\theta}$$

turn $\Delta\theta$ into $d\theta$

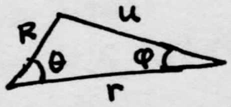
$$= 2\pi G\rho m \int_0^{\pi} \frac{R^2 \sin\theta \cos\varphi \, d\theta}{u^2}$$

Now we need some trickery to eval this:

Law of Cos



$a^2 = b^2 + c^2 - 2bc\cos\alpha$



It turns out we want

(★) $R^2 = r^2 + u^2 - 2ru\cos\varphi$

(★★) $u^2 = R^2 + r^2 - 2Rr\cos\theta$

differentiate (★★): $2u\,du = -2Rr(-\sin\theta)\,d\theta$

$\Rightarrow \frac{u}{r}\,du = R\sin\theta\,d\theta$

$$= 2\pi \, G\rho m \int_{\theta=0}^{\pi} \frac{R\cos\varphi}{u^2} \, \frac{u}{r} \, du$$

Now change limits of integration:

(★★) $u = \pm\sqrt{R^2 + r^2 - 2Rr\cos\theta}$

distance

$\theta = 0 \Rightarrow u = \sqrt{R^2 + r^2 - Rr} = \sqrt{(R-r)^2} \Rightarrow u = \begin{cases} r-R & r>R \\ R-r & r<R \end{cases}$

if p outside / p inside

$\theta = \pi \Rightarrow u = \sqrt{R^2 + r^2 + Rr} \Longrightarrow u = R + r$

$$= 2\pi G\rho m \int_{u=A}^{R+r} \frac{R\cos\varphi}{ur} \, du$$

A = r−R outside
R−r inside

$$= 2\pi G\rho m \int_{u=A}^{R+r} \frac{R}{2}\left[\frac{1}{r^2} + \frac{1}{u^2}\left(1 - \frac{R^2}{r^2}\right)\right] du$$ ⟵ using (★) $\frac{-R^2 + r^2 + u^2}{2ru} = \cos\varphi$

$\frac{R}{ur}[\cos\varphi] = \frac{R}{2ur}\left[\frac{r}{u} + \frac{u}{r} - \frac{R^2}{ur}\right]$

$$= \cancel{2}\pi G\rho m \frac{R}{\cancel{2}}\left[\frac{1}{r^2}u - \left(1 - \frac{R^2}{r^2}\right)\frac{1}{u}\right]_A^{R+r}$$

let $P := R + r$, $M := R - r$ then p inside ⇒ $A = M$, outside ⇒ $A = -M$

$$= \frac{\pi G\rho m R}{r^2}\left[P - A - MP\left(\frac{1}{A} - \frac{1}{P}\right)\right]$$

outside $A = -M$:

$\frac{\pi G\rho m R}{r^2}\underbrace{\left[P + M + MP\left(\frac{1}{M} + \frac{1}{P}\right)\right]}_{= 2(P+M) \, = \, 4R}$

$= \frac{G(\rho 4\pi R^2)m}{r^2} = \boxed{\frac{GMm}{r^2}}$ just like for a pt mass!

area of sphere: $4\pi R^2$

$\boxed{F = -\frac{GMm}{r^2}\hat{e}_r}$

Inside $A = M$

$\left[P - M - MP\left(\frac{1}{M} - \frac{1}{P}\right)\right]$

$P - M - (P - M)$

$= 0$ for p anywhere inside.

□

Cont'd ⟶

For a pt. on the ring K_θ, we know $V = \frac{-G(\Delta M_{\alpha,\theta})m}{r}$ and $\vec{F} = -\nabla V$

Thus we can repeat our procedure (the only difference is now we have r^{-1} vs. r^{-2})

No concern for symm now!

$$\Delta M_{\alpha,\theta} = \rho\,(R\sin\theta\,\Delta\alpha)(R\Delta\theta)$$

then $$V_{K_\theta} = -\int_{\alpha=-\pi}^{\pi} \frac{G\,(\Delta M_{\alpha,\theta})\,m}{u}\,d\alpha$$

They put − sign

$$= \frac{-G\rho R^2 \sin\theta\,\Delta\theta\, 2\pi}{u}$$

$$V = -2\pi G\rho m \int_{\theta=0}^{\pi} \frac{R^2 \sin\theta\, d\theta}{u}$$

Subs like before
$\frac{u}{r}du = R\sin\theta\, d\theta$
limits of integration are the same too:
$\theta = 0 \Rightarrow u = A$
$\theta = \pi \Rightarrow u = R+r$

$$= -m\, 2\pi R G\rho \int_{u=A}^{R+r} \frac{1}{\cancel{u}}\frac{\cancel{u}}{r}\,du$$

$$= \frac{-m\, 2\pi R G\rho}{r}\left[R+r-A\right]$$

Outside

Just like before $A = -M$
$= r - R$

$R + r - (r - R)$
$= 2R$

$$\Rightarrow \frac{-2\pi\, m G R\rho\,(2R)}{r}$$

$$= \frac{-G\,m\,(\rho\, 4\pi R^2)}{r}$$

$V(r) = \frac{-GmM}{r}$ for any pt outside.

same potential as for a pt. mass!

Inside

$A = M = R - r$

$$\Rightarrow \frac{-m\, 2\pi R G\rho\,(2\cancel{r})}{\cancel{r}}$$

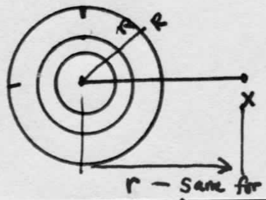
$= -mG\rho(4\pi R)$ const $\quad -Gm\rho\, 4\pi R\frac{R}{R} = \frac{-GmM}{R}$ (const)

$V(r) = \text{const}$ for any pt inside.

$\Rightarrow F = -\nabla V = 0 \quad \square$

[See Griffiths ITE p. 85-86 for another version of this derivation for electrostatics (essentially same arg)]

▷ What is the potential outside a solid ball $B(0,R)$?



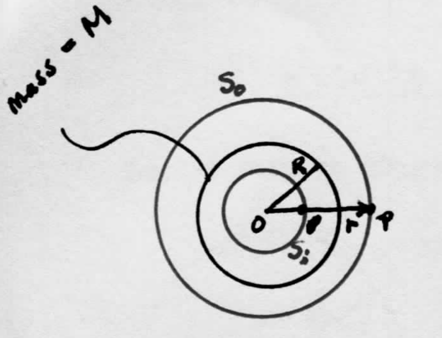
$$V_B(x) = \int_{\text{all } S} V_S(x) \qquad V_{S(0,R)} = \frac{-Gm}{r}dM \qquad dM = \underbrace{\rho}_{\text{density}} \cdot \underbrace{4\pi R^2 dR}_{\text{vol}}$$

$$V_B(x) = \int_{R=0}^{R} \frac{-Gm}{\underset{\text{const}}{r}}(\rho\, 4\pi R^2\, dR) = \frac{-Gm}{r}\left(\rho\,\frac{4\pi}{3}R^3\right) = \boxed{\frac{-GmM}{r}}$$

Same as for a pt as expected $\square$

For potential inside see Feynman E&M ch 4 sheets

Now derive these results very easily from Gauss' Law for Gravity



From Marsden & Tromba VC ch 7.4 #16 we can use Gauss' Law for Gravity for a sphere (due to the symm)

Just identify mass M with pos charge Q.
Then Gauss becomes $\int_S \vec{F}_G \cdot \hat{n}\, dA = M_{inside}$

From S^2-symm, we know $\vec{F} = F(r)\hat{e}_r$; only dependence on r, homogeneous wrt θ, φ. spherical co-ord angles.

▷ Outside

Now consider a larger concentric sphere S_o that touches p. $S_o = S^2_{(o,r)}$

Gauss: $M_{inside} = \int_{S_o} F(r)\hat{e}_r \cdot \hat{e}_r\, dA = F(r)\int_0^{2\pi}\int_0^{\pi} 1 \cdot r^2 \sin\varphi\, d\theta\, d\varphi = F(r)\, r^2 4\pi$

$$\Rightarrow \boxed{\frac{1}{4\pi}\frac{M}{r^2} = F(r)}$$

now consider another mass, test mass m at position vector $\vec{r} = r\hat{e}_r$. By linearity of $\vec{F}_G$

Force on m due to sphere. $\boxed{F = \frac{1}{4\pi}\frac{mM}{r^2}}$ M&T VC ch 7.3 #16

▷ Inside

Same idea, now $r<R$
No mass inside

$$0 = \int_{S_i} F(r)\hat{e}_r \cdot \hat{e}_r\, dA \Rightarrow 0 = F(r)\, r^2 4\pi$$

$$\Rightarrow \boxed{F(r) = 0 \quad \text{for any } r<R}$$

▷ Does Gauss' Law show that the grav field inside an arb. empty cavity is 0, like it does for electrostatic field in a conductor?

No!
a body of mass does not correspond to a conductor, it would correspond to a solid body of pos charge. In electrostatics, the key idea is that the E field must be 0 in a conductor since no charge is flowing, and this makes $E=0$ in any cavity.

Fowles ch 6 Rewrite

ch 6.[redacted]4 Potential Energy in a General Cental Field

a general isotropic central field $\vec{F} = f(r)\hat{e}_r$

To show it is Conservative, we show $\nabla \times F = 0$ Lets use $\nabla \times (\cdot)$ in spherical co-ords

$$\nabla \times F = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\, \hat{e}_\varphi \\ D_r & D_\theta & D_\varphi \\ F^{r} & rF^{\theta} & r\sin\theta F^{\varphi} \end{vmatrix}$$

where $F^r = f(r)$ no θ, φ dependence
$F^\theta = 0$
$F^\varphi = 0$

$$= \nabla \times F = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{e}_\theta + -\frac{1}{r} f_\theta \hat{e}_\varphi = 0$$

Since F conservative, we can define a PE fcn (path indep)

M&T VC
Baby Poincare Lemma

$$\boxed{V(r) := -\int_{r_{ref}}^{r} \vec{F}\cdot d\vec{r} = -\int_{r_{ref}}^{r} f(r)\,dr}$$

r_{ref} ← define PE=0 here
often $r_{ref} = \infty$

Then $f(r) = -\frac{d}{dr}V(r)$

ch 6.5 Angular Momentum in Central Field

we will show that $\vec{L} = \text{const}$ and thus a particle ~~is constrained to~~ always remains in a plane.

J. Rosen makes a Symm arg to justify this ASPFS p.

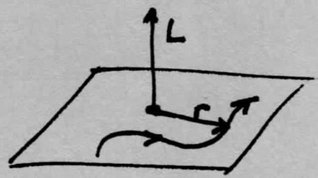
$$\boxed{\text{Thm} \quad \text{Central force} \Rightarrow \vec{L} = \text{const wrt } t \quad \text{a particle always remains in same plane}}$$

$\vec{F} = f(r)\hat{e}_r = f(r)\hat{r}$

$m\ddot{\vec{r}} = f(r)\hat{r}$

pf. [redacted] Ang momentum of particle wrt origin $\vec{L} = \vec{r}\times m\vec{v} = \vec{r}\times m\dot{\vec{r}}$

$$\frac{d}{dt}\vec{L} = \dot{\vec{r}}\times m\dot{\vec{r}} + \vec{r}\times m\ddot{\vec{r}}$$
$$= 0 + \cancel{\vec{r}\times m\frac{f(r)}{r}\vec{r}} = 0$$



$\Rightarrow \vec{L} = \text{const.}$

$\vec{r}(t)$ remains in the plane $\{\vec{L}\}^\perp$ because for $c = a\times b$ $c \perp a$ and $c \perp b$ □

Since we are in a plane, we can employ polar co-ords.

$\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$ ch 1

$$L = \|\vec{r}\times m\vec{v}\| = \|r\hat{e}_r \times m(\dot{r}\hat{e}_r + r\dot{\theta}e_\theta)\| = \|r^2 m\dot{\theta}\, e_r\times e_\theta\| = |r^2 m\dot{\theta}|\underbrace{\|e_r\times e_\theta\|}_{=1}$$

$L = m\,|r^2\dot{\theta}|$ → call this $|h|$ $\quad L = m|h|$

and we know this is const wrt t.

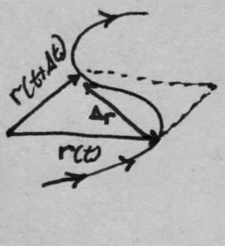
ch 6.6 Law of Areas. Kepler's Laws of Planetary motion.

Thm ~~[illegible]~~
Let $\vec{r}(t)$ be position vect of particle on any planar traj
Let $A(t)$ be area swept out by $\vec{r}$ wrt some ref time t_0
$\Rightarrow \frac{dA}{dt} = \frac{L(t)}{2m}$

$\frac{dA}{dt} = \frac{L}{2m}$ = const for central force motion

This establishes Kepler II

pf. area of parallelogram $\|r\times(r+\Delta r)\|_2 = \|r\times\Delta r\|_2$ we only want the left half $A = \frac{1}{2}\|r\times\Delta r\|$



write ΔA as we are working with incr Δr and divide both sides by Δt:

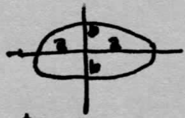
$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\left\| r\times\frac{\Delta r}{\Delta t}\right\| \quad \text{take} \lim_{\Delta t\to 0} \quad \frac{dA}{dt} = \frac{1}{2}\|\vec{r}\times\dot{\vec{r}}\|$$
$$= \frac{1}{2}\frac{1}{m}\|r\times mv\| = \frac{L(t)}{2m}$$

and L = const for central force □

Kepler's Laws I) Earth planet moves in ellipse w/ Sun at one focus,

II) Radius vector sweeps out equal area in equal times

III) $\tau^2 = (\text{const})\, a^3$



τ = period of revolution

ch 6.7 Orbit of Particle in Central-Force Field

$$m\ddot{\vec{r}} = f(r)\,\hat{e}_r \quad \text{in plane}$$

using ch 1 accel in polar $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

so $m(\ddot{r} - r\dot{\theta}^2) \overset{!}{=} f(r)$ ★

$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \overset{!}{=} 0$ ★★

observe $\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta}$

$= r(2\dot{r}\dot{\theta} + r\ddot{\theta})$

$= 0$ by ★★

So we recover $r^2\dot{\theta} = \text{const} =: h$ from prev page

To find the shape of the orbits (It is harder to find explicit time dependence $\vec{r}(t)$) → we will solve for $r(\theta)$

Make COV $r = \frac{1}{u}$ in ★ $\quad m(\ddot{r} - r\dot{\theta}^2) = f(r)$

$$\dot{r} = -u^{-2}\frac{du}{dt} = \frac{-1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = \underbrace{\frac{-1}{u^2}\dot{\theta}}_{-r^2\dot{\theta}}\frac{du}{d\theta} = -h\frac{du}{d\theta}$$

$$h = \frac{1}{u^2}\dot{\theta} \qquad \dot{\theta} = u^2 h$$

$$\ddot{r} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h(u^2h)\frac{d^2u}{d\theta^2}$$

$$m(\ddot{r} - r\dot{\theta}^2) = f(r)$$

$$m\left(-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}\underbrace{\left(u^2 h\right)^2}_{u^4h^2}\right) = f(1/u)$$

$$-mh^2u^2\left(u''(\theta) + u\right) = f(1/u)$$

$$\boxed{u''(\theta) + u(\theta) = \frac{-1}{mh^2u^2}f(1/u)}$$

ex 6.3 Given a particle in a central field in spiral orbit $r(\theta) = c\theta^2$, what is $\vec{F}$?

$u = \frac{1}{c}\theta^{-2} \;\Big|\; \frac{du}{d\theta} = \frac{-2}{c}\theta^{-3} \;\Big|\; \frac{d^2u}{d\theta^2} = \frac{6}{c}\theta^{-4} = 6cu^2$

Plug in:

$$6cu^2 + u \overset{!}{=} \frac{-1}{mh^2u^2}f(1/u) \Rightarrow f(1/u) = -mh^2u^2(6cu^2 + u)$$

$$= -mh^2(6cu^4 + u^3)$$

$$\Rightarrow f(r) = -mh^2\left(\frac{6c}{r^4} + \frac{1}{r^3}\right)$$

ex 6.4 How does θ vary with t?

Const $h = r^2\dot{\theta} \Rightarrow \dot{\theta} = h\frac{1}{r^2} = \frac{h}{c^2\theta^4}$

IC $\theta(0) = 0 \qquad \int \theta^4\,d\theta = \int \frac{h}{c^2}\,dt$

$\frac{1}{5}\theta^5 = \frac{h}{c^2}t \Rightarrow \theta(t) = \left(\frac{5h}{c^2}\right)^{1/5} t^{1/5}$ □

ch 6.7 cont'd

Energy eq of the orbit

Conservation of Energy means Const $E_0 = KE + PE$

$$= \frac{1}{2}mv^2 + V(r)$$

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \qquad = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$\vec{v}\cdot\vec{v} = \dot{r}^2 + r^2\dot{\theta}^2$$

we know $\left.\begin{aligned}\dot{r} &= -h\frac{du}{d\theta}\\ \dot{\theta} &= hu^2\end{aligned}\right\} \longrightarrow = \frac{m}{2}\left[h^2\left(\frac{du}{d\theta}\right)^2 + \frac{1}{\cancel{u^2}}h^2 u^{\cancel{4}\,2}\right] + V(r)$

Energy eq of orbit $\qquad \boxed{E_0 = \frac{mh^2}{2}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + V(1/u)}$

ex 6.5 continuing prev examples

$$r = c\theta^2 \Rightarrow u = \frac{1}{c\theta^2} \qquad \frac{du}{d\theta} = \frac{-2}{c\theta^3}$$

$$E_0 = \frac{mh^2}{2}\left[\left(\frac{-2}{c\theta^3}\right)^2 + \left(\frac{1}{c\theta^2}\right)^2\right] + V(1/u) = \frac{mh^2}{2}\left[\frac{4}{c^2\theta^6} + \frac{1}{c^2\theta^4}\right] + V(1/u)$$

$$= \frac{mh^2}{2}\left[\frac{4c}{r^3} + \frac{1}{r^2}\right] + V(r)$$

Thus $\quad V(r) = E_0 - \frac{mh^2}{2}\left[\frac{4c}{r^3} + \frac{1}{r^2}\right] \quad$ and $\quad f(r) = \frac{-d}{dr}V(r) \qquad \square$

ch 6.8 Orbits in an Inverse-Square Field