Fowles ch 6 Summary

I should also add in material from Marion CDOPAS, Roy OM ①

Force on particle i due to particle j

p. 134 $\quad \vec{F}_{ij} = \dfrac{G m_i m_j}{r_{ij}^3} \vec{r}_{ij} \qquad F_{ij} = -F_{ji}$

$$G :\doteq 6.672 \times 10^{-11} \frac{N\, m^2}{kg^2}$$

union of concentric shells S^2

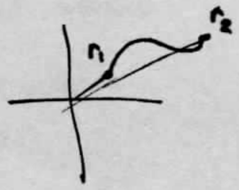
6.2 Gravitational Force Between a uniform Sphere and a Particle.

Spherical shell: centered at origin

M = mass of shell or concentric shell bundle.

$$\vec{F} = \begin{cases} -\dfrac{GMm}{r^2}\hat{e}_r & \text{particle } \underline{\text{outside}} \\ 0 & \text{particle } \underline{\text{inside}} \end{cases}$$

6.3 Gravitational Potential



$$W = \int_C \vec{F}\cdot d\vec{s} = GMm \int_{r_1}^{r_2} \frac{1}{r^2}\,dr = -GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

Take reference pt at ∞ and get:

$$V(r) := GMm \int_{\infty}^{r} \frac{1}{r^2}\,dr = -\frac{GMm}{r}$$

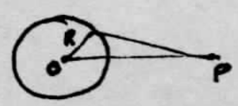
Divide by m to get grav potential per unit mass Φ

$$\vec{F} = -\nabla V$$

(6.1) Potential Fcn of Unif spherical shell

$$\Phi(r) = -\frac{GM}{r}$$

(6.2) Potential of circular ring (in same plane as ring)



$$\Phi(r) = -\frac{GM}{r}\left(1 + \frac{R^2}{4r^2} + \dots\right)$$

6.4 Potential Energy in a General Central Field

$\vec{F} = f(r)\,\hat{e}_r$ then $\nabla \times F = 0$ (use curl in spherical co-ords)

Define $V(r) := -\displaystyle\int_{\vec{r}_0}^{\vec{r}} \vec{F}\cdot d\vec{s} = -\int_{r_0}^{r} f(r)\,dr$

6.5 Angular Momentum in Central Fields

$\vec{L} := \vec{r} \times \vec{p}$ want to show: $\vec{L}$ = const for central force and thus motion is confined to a single plane.

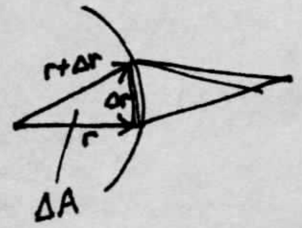
$\dot{L} = \underbrace{\dot{r} \times \vec{p}}_{0} + \underbrace{r \times \dot{p}}_{r \times F}$ central Force $\Rightarrow \vec{F} = f(r)\hat{e}_r - f(r)\,\hat{r}$

$\dot{L} = 0 \Rightarrow \vec{L}$ = const

Introduce polar co-ords $\quad \|L\| = |m r^2 \dot{\theta}|$ = const $\quad$ we define $h := |r^2 \dot{\theta}|$

6.6 Law of Areas. Kepler's Laws of Planetary Motion

For any central force, the rate of area being swept out by the position vector of a moving particle is const: $\dot{A} = \frac{L}{2m}$



$$\Delta A \approx \tfrac{1}{2}\|(\vec{r})\times(\vec{r}+\vec{\Delta r})\| = \tfrac{1}{2}\|r\times\Delta r\|$$

Divide by Δt and take $\lim \Delta t \to 0$

$$\frac{\Delta A}{\Delta t} = \tfrac{1}{2}\|r\times\frac{\Delta r}{\Delta t}\| \to \dot{A} = \tfrac{1}{2}\|r\times v\| = \frac{1}{2m}\|r\times mv\|$$

established here

KEPLER: (i) Each planet moves in an ellipse with sun at one focus.
(ii) Radius vector sweeps out equal areas in equal times
(iii) $\tau^2 = c\,a^3$ where τ = orbital period
a = semi-major axis of ellipse
c = const which depends on M and m.

6.7 Orbit of a Particle in a Central Force Field

$\vec{a} = (\ddot{r}-r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta}+2\dot{r}\dot{\theta})\hat{e}_\theta$

$\Rightarrow m(\ddot{r}-r\dot{\theta}^2) = f(r) = \frac{-k}{r^2}$

$m(2\dot{r}\dot{\theta}+r\ddot{\theta}) = 0$

Force emanating from the origin ⊛ $\boxed{m\ddot{\vec{r}} = f(r)\,\hat{e}_r}$ in polar co-ords

Let us try to solve for r as a fcn of θ (rather than the complete solution $(r(t), \theta(t))$.)

Define $u := \frac{1}{r}$ and recall that $h = r^2\dot{\theta}$ = const

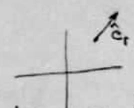
$$\dot{r} = -u^{-2}\dot{u} = -u^{-2}\frac{du}{d\theta}\frac{d\theta}{dt} = -r^2\dot{\theta}\frac{du}{d\theta} = -h\,u'(\theta)$$

$$\ddot{r} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h^2u^2u''$$

Then ⊛ becomes: $u'' + u = \frac{1}{mh^2u^2} f(u^{-1})$

Energy Eq of orbit: Central force field is Conservative $\Rightarrow$ Energy is conserved

$$E = KE + PE$$
$$= \tfrac{1}{2}mv^2 + V(r)$$
$$= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \text{const.}$$



6.8 Orbits in an Inverse-Square Field

minus sign because $\hat{e}_r$ is directed away from the origin, and we want the force pointing toward origin.

Now we specialize $f(r) = \dfrac{-k}{r^2}$ where $k = GMm$

[For 2 body problem of p. 175, $k = G(M+m)m$]

Then the ODE $u'' + u = \dfrac{k}{mh^2}$

has the solution $u(\theta) = A\cos(\theta - \theta_0) + \dfrac{k}{mh^2}$ Take $\theta_0 = 0$

$$r(\theta) = \frac{1}{A\cos\theta + k/mh^2}$$

or

Polar Eq of conic section with origin at one focus

$$\boxed{r(\theta) = \frac{r_0(1+e)}{1 + e\cos\theta}}$$

where $e := \dfrac{Amh^2}{k}$

$$r_0 := r(0) = \frac{mh^2}{k(1+e)}$$

eccentricity

circle ↓ ellipse Parabola ↓ hyperbola

0 1 e

$$r_{min} = r(0)$$
$$r_{max} = r(\pi) = r(0)\frac{(1+e)}{(1-e)}$$

Orbital Parameters from the Conditions at Closest Approach

we want to find a special velocity V_c such that if the particle has this velocity at r_{min}, then the orbit is circular.

solve for e in relation for r_0: $\quad e = \dfrac{mh^2}{kr_0} - 1 = \dfrac{mr_0V_0^2}{k} - 1$

set $e = 0$ and solve for V_0 (V_c):

$$V_c = \sqrt{\frac{k}{mr_0}} = \sqrt{\frac{GM}{r_0}}$$

$r_{min} = r(0) = r_0$

$V(0) = V_0$

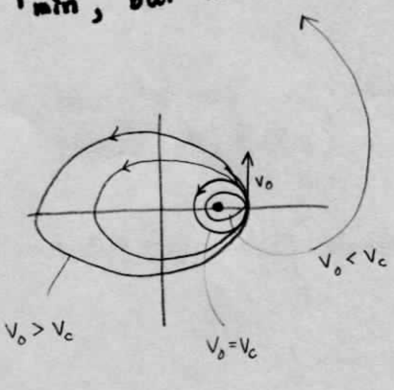
Then for any orbit $\quad e = \dfrac{V_0^2}{V_c^2} - 1$

but then we must have $V_0 \geq V_c$ or else e is neg and that doesn't make sense in the conic eq.

[if $V_0 < V_c$, then we are not at the peri r_{min}, but rather r_{max}.]

NOTE THAT $\vec{V}$ is perpendicular to $\vec{r}$ only at r_{min} & r_{max}

$$\boxed{r(\theta) = \frac{(V_0/V_c)^2}{1 + \left[\left(\frac{V_0}{V_c}\right)^2 - 1\right]\cos\theta}}$$



so a given r defines a minimum perpendicular velocity $V_c(r)$ to have a peri [$r_{min} = r$]

6.9 Orbital Energies in the Inverse-Square Field

$-\frac{k}{r}$

For potential $V(r) = -\frac{k}{r}$ we have conservation of energy eq $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$

With the substitution $u = \frac{1}{r}$

$$\frac{1}{2}mh^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - ku = E \qquad (6.27)\ \text{p.146}$$

we solve this for θ as a fcn of u and obtain $\cos\theta = \dfrac{mh^2u - k}{\sqrt{k^2 + 2Emh^2}}$

solving for u and using $r = 1/u$ we obtain the results:

making explicit the dependence on energy E

① $$r = \frac{\frac{mh^2}{k}}{1 + \sqrt{1 + \frac{2Emh^2}{k^2}}\cos\theta} \stackrel{!}{=} \frac{r_0(1+e)}{1 + e\cos\theta}$$

② $$e = \sqrt{1 + \frac{2Emh^2}{k}}$$

Potential Depends on $V(\infty) = 0$

③ $E < 0 \iff -1 < e < 1$ but we only care about $e \geq 0$

$E = 0 \iff e = \pm 1$

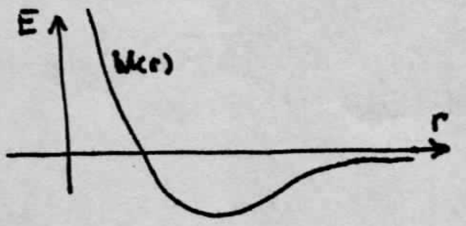
$E > 0 \iff e > 1$ (or $e < -1$)

6.10 Limits of the Radial Motion. Effective Potential

We want to write the energy eq of the orbit as form " $\boxed{\frac{1}{2}m\dot{x}^2 + V(x) = E_0}$ ", but for x we want the radial dist r. [one-dim problem w/ a potential]

we have (6.26) $\frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E$ and recall $h = r^2\dot{\theta}$

$$\Rightarrow \frac{m}{2}\dot{r}^2 + \underbrace{\frac{m}{2}\frac{h^2}{r^2} + V(r)}_{U(r)\ \text{Effective potential}} = E_0$$



$$\boxed{\frac{m}{2}\dot{r}^2 + U(r) = E_0}$$

solve for $\dot{r}$: $\dot{r} = \pm\sqrt{\frac{2}{m}(E_0 - U(r))}$ Thus $\dot{r} = 0$ at such values of r that $E_0 = U(r)$

Now if $U(r_0) = E$

$\Rightarrow \dot{r} = 0$ The only orbit for which the radius doesn't change is a circle.

See Marion CDOPAS p. 255

ASIDE Escape Velocity (1-dim motion straight up)

This is Fowles p. 45

$\frac{1}{2} m \dot{x}^2 + V(x) = E$

E is const

Thus $\frac{m}{2} v_0^2 + V(0) =: E_0$

$V(x) = \frac{-GMm}{R+x}$

$R + x = r$

R = radius of planet (say Earth)

we write it this way so no division by 0

$V(0) = -\frac{GMm}{R}$

x is height above planet's surface; rename it h

For a given v_0, how high do we go?

The eq is:

$$\frac{m}{2}\cancel{v^2} - \frac{GMm}{R+h} = \underbrace{\frac{m}{2} v_0^2 - \frac{GMm}{R}}_{E_0} \quad ⊛$$

at max height → 0

Solve for h as a fcn of v_0:

$$\boxed{h(v_0) = \frac{-v_0^2 R^2}{v_0^2 R - 2GM}}$$

To find escape velocity, set $h = \infty$ in ⊛ and solve for v_0:

$$0 = \frac{v_0^2}{2} - \frac{GM}{R} \Rightarrow \boxed{v_0 = \sqrt{2\frac{GM}{R}} \text{ for escape}}$$

Note that the "g" of gravity at the Earth's surface (ie mg) is $g = \frac{GM}{R}$.

6.11 Periodic Time of Orbital Motion

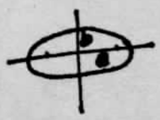
we want to derive Kepler's 3rd law $\tau = 2\pi \left(\frac{m}{k}\right)^{1/2} a^{3/2}$

τ = orbital period

a = semi-major axis of ellipse

In the real 2 Body problem $k = G(M+m)m$ p. 175

$\frac{dA}{dt} = \frac{h}{2}$ where we assume sign is pos.



$\Rightarrow A(t) = \frac{h}{2} t$

But area of ellipse is $\pi a b$

$A(\tau) = \pi a b$

$\Rightarrow \tau = \frac{2}{h} \pi a b$

From geometry of ellipse $b = a\sqrt{1-e^2}$

also $2a = r(0) + r(\pi)$

After some tricky substitutions, the result follows.