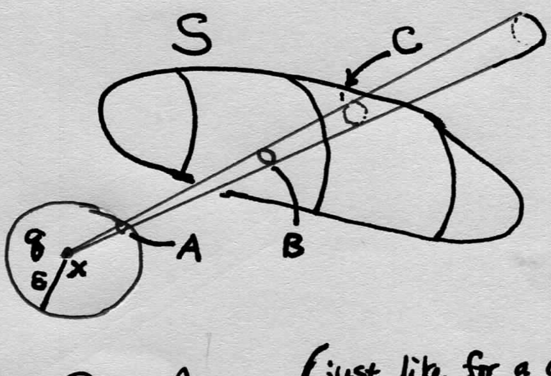
Gauss' Law $\int_S \vec{E}\cdot\hat{n}\,dA = \frac{1}{\epsilon_0}Q_{inside}$ Justified using solid angle

Here we restrict to nice surfaces like an ellipsoid. We just want to illustate the method, not deal with a lot of complex cases.

Charge outside We must show the flux going in equals flux coming out (so no accumulation) Of course this is true for rays, but we care about the strength over the surf area.



Consider a single charge of value q outside S at pt x. Consider a cone of rays emanating from q and cutting out a region B where it enters S, and a region C where it exits.

Let there be a small sphere $S^2(x,\varepsilon)$ not touching S and the cone cuts area A in it.

We know the solid angle Ω cut by the cone is $\Omega = \frac{A}{\varepsilon^2}$ (just like for a circle $\theta = \frac{s}{r}$)

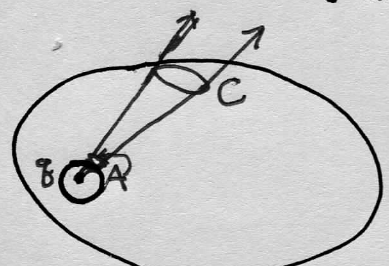
From the $\boxed{\frac{1}{r^2}\text{ Flux Cone / Solid Angle Thm}}$ REA prob 19-50 – also written up in M&T sheets ch 2.4

A & B are caps on the cone $\Rightarrow$ Flux thru B = solid angle at A = Ω

A & C are caps on the cone $\Rightarrow$ Flux thru C = Ω

$\Rightarrow$ all flux that enters at B also leaves at C – no accumulation $\square$

Charge Inside We must show the full flux from the charge all passes out thru S (none decays by dist as would be the case with $F = \frac{1}{r^2}e^{-\lambda r}$ for example)



Cone can be a more general shape than circular

Now tiny sphere $S^2(x,\varepsilon)$ is inside

Full Solid angle of sphere is $\Omega = 4\pi$ and Coulomb flux is $\Phi = q\Omega = 4\pi q$ in Gaussian units

Make a fine mesh grid over the sphere

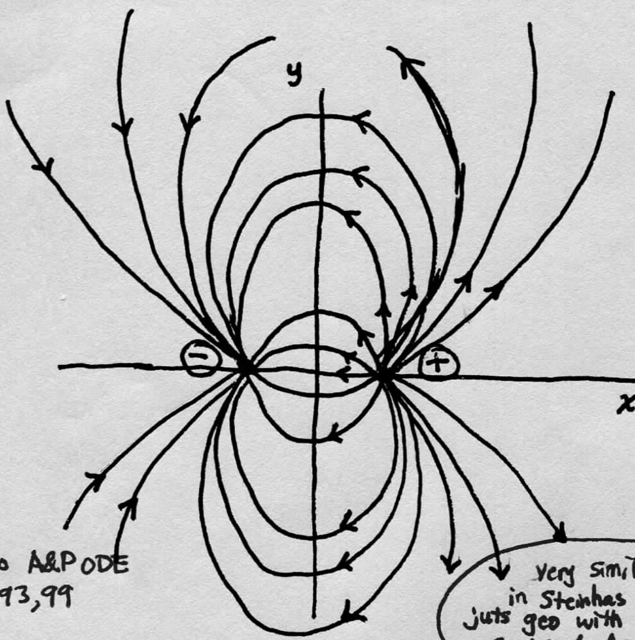
Thru each box A there is a radial cone cutting out B on the inside of Surf S.

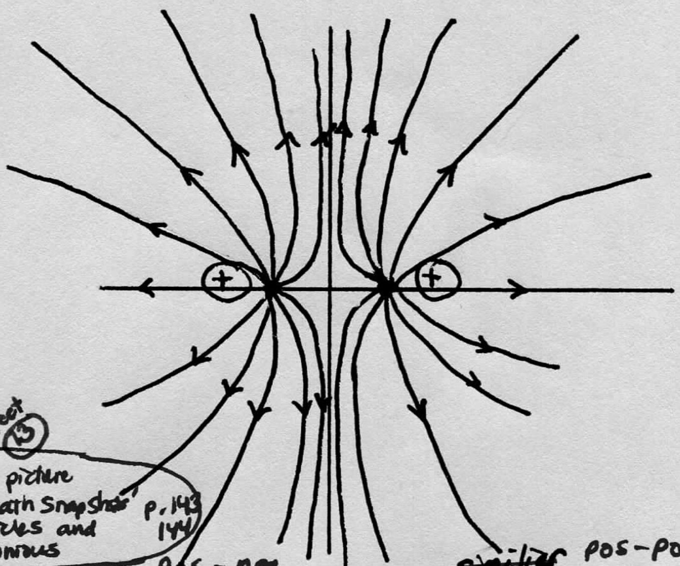
We know the flux is the same thru each cap

All the boxes make solid angle 4π (the whole sphere)

So the outward flux thru S is $4\pi q$

$\square$





Maxuell p. 182–184 draws this by some other method of intersecting lines

I took these from Smythe p. 9, 10

Also A&PODE p. 93, 99

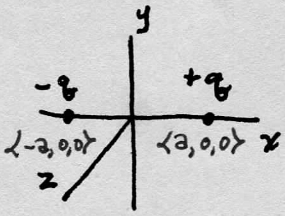
Sheet 13

Very similar picture in Steinhas 'Math Snapshots' p. 143, 144 juts geo with circles and circles of Apollonius

Every book shows these diagrams of 2 ~~opposite~~ pos – neg and 2 ~~same~~ similar pos–pos charges, but how do we calculate these $\vec{E}$ flow lines? ('Field lines', 'Lines of force').

Consider 2 charges on a line, the x axis. (Later we shall add more charges)

<u>Step 1</u> Reduction by Sym – we will give a Sym arg to show we only need to consider the E field in the xy plane.

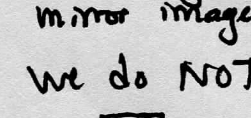


(1) This config has reflection sym across the xy plane.
This means at any pt $\langle x,y,0\rangle$ $\vec{E}$ can have no component $\perp$ to xy plane. If $E^{\perp}$ was not 0 $E^{\perp}$ would be reflected to $-E^{\perp}$ and this would make a contradiction. [An observer $\mathcal{O}_1$ in the original co-ords would see a pos $E^{\perp}$ say, but an equally valid observer $\mathcal{O}_2$ in co-ords with z axis reflected would see and measure the config identically but at $\langle x,y,0\rangle$ $E^{\perp}$ would be neg. $\Rightarrow\Leftarrow$] See ch 5 writeup

(2) The config has rotation sym about the x axis.
This means the whole E vf can be obtained by rotating the xy plane by an angle 2π around x axis. This also includes a 3rd sym

(3) The config has reflection sym across xz plane, so ~~top~~ bot half in xy plane is mirror image of the ~~bot~~ top half.

(4) We do NOT have reflection sym about an arb plane $\perp$ to x axis



That would force $\vec{E}$ to be radial, as in ch 5 with infite charged cylinder.

We do have reflect sym about the midpt plane for 2 charges of same sign. [for charges with opp sign, we can still see sym in the flowlines, but this is apparent only when we have solved the problem — the config does not have this sym]

$\Rightarrow$ So we know we only need to consider the field E and the assoc flow lines in the upper half of xy plane.

Probably better is $\dfrac{dy/dt}{dx/dt} = \dfrac{E^2}{E^1}$ for pts where $E^1(x) \neq 0$

$\Rightarrow E^2 \dfrac{dx}{dt} - E^1 \dfrac{dy}{dt} = 0$

Step 2 set up ODE $\qquad \dot{x} = E(x) \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} E^{(1)}(x) \\ E^{(2)}(x) \\ E^{(3)}(x) \end{bmatrix}$

Smythe p.7

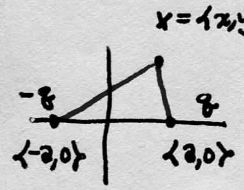
and we restrict to x,y plane $\qquad E^{(3)}(x,y,0) = 0$

so we have $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} E^1(x) \\ E^2(x) \end{bmatrix}$

$$\left.\begin{aligned} \frac{dx}{dt} &= E^1 \\ \frac{dy}{dt} &= E^2 \end{aligned}\right\} \Rightarrow \frac{dx}{E^1} = dt = \frac{dy}{E^2}$$

$$(*) \qquad \Rightarrow E^{(2)} dx - E^{(1)} dy = 0$$

[I won't work with dy/dx because from the known soln curves in the plot, y is not always a fcn of x]



we will write $\vec{x} - \vec{a}$ for $\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ 0 \end{bmatrix}$

Coulomb's Law $\qquad \vec{E}(x) = \dfrac{kq}{\|x-a\|^3}(\vec{x}-\vec{a}) + \dfrac{k(-q)}{\|x--a\|^3}(\vec{x}+\vec{a})$

$$\frac{1}{kq} E^{(1)} = \frac{(x-a)}{[(x-a)^2+y^2]^{3/2}} - \frac{(x+a)}{[(x+a)^2+y^2]^{3/2}}$$

Call this $N^{3/2}$ (it has neg sign) $\qquad P^{3/2}$

$$\frac{1}{kq} E^{(2)} = \frac{y}{N^{3/2}} - \frac{y}{P^{3/2}}$$

Step 3 Demonstrate that $\boxed{C(x,y) := \dfrac{(x+a)}{P^{1/2}} - \dfrac{(x-a)}{N^{1/2}}} = c$ is a 1st Integral or Const of Motion on a particular soln curve

Smythe p.8 attempts to "derive" this with a lot of trickery that seems dubious. Better to just verify it. [Next we will give another derivation from Gauss' Law]

we don't know this yet $\quad C_x dx + C_y dy = 0$ because

a priori we can differentiate C to get $C_x dx + C_y dy$ [redacted]

If we can show $C_x = E^{(2)}$ and $C_y = -E^{(1)}$ then we have it: $E^{(2)} dx - E^{(1)} dy = 0$

$$\begin{aligned} C_x &= P^{-1/2} + (x+a)\left(\tfrac{-1}{2} P^{-3/2} 2(x+a)\right) - \left[N^{-1/2} + (x-a)\left(\tfrac{-1}{2} N^{-3/2} 2(x-a)\right)\right] \\ &= P^{-1/2} - (x+a)^2 P^{-3/2} - N^{-1/2} + (x-a)^2 N^{-3/2} \\ &= \frac{P}{P}\frac{1}{P^{1/2}} - \frac{(x+a)^2}{P^{3/2}} - \frac{N}{N}\frac{1}{N^{1/2}} + \frac{(x-a)^2}{N^{3/2}} \\ &= \frac{(x+a)^2 + y^2 - (x+a)^2}{P^{3/2}} - \left[\frac{(x-a)^2 + y^2 - (x-a)^2}{N^{3/2}}\right] \\ &= \frac{y^2}{P^{3/2}} - \frac{y^2}{N^{3/2}} \end{aligned}$$

Similarly:

$$C_y = (x+a)\left(\tfrac{-1}{2} P^{-3/2} 2y\right) - (x-a)\left(\tfrac{-1}{2} P^{-3/2} 2y\right) = \frac{-(x+a)y}{P^{3/2}} + \frac{(x-a)y}{N^{3/2}}$$

$$C_x\,dx + C_y\,dy \overset{?}{=} 0$$

$$\Rightarrow \left(\frac{y^2}{P^{3/2}} - \frac{y^2}{N^{3/2}}\right)dx + \left(\frac{(x-a)}{N^{3/2}}y - \frac{(x+a)}{P^{3/2}}y\right)dy \overset{?}{=} 0$$

$$= -y\left[\left[\frac{-y}{P^{3/2}} + \frac{y}{N^{3/2}}\right]dx + \left[-\frac{(x-a)}{N^{3/2}} + \frac{(x+a)}{P^{3/2}}\right]dy\right]$$

$$= -y\left[\left(\frac{y}{N^{3/2}} - \frac{y}{P^{3/2}}\right)dx - \left(\frac{(x-a)}{N^{3/2}} - \frac{(x+a)}{P^{3/2}}\right)dy\right]$$

$$= \frac{-y}{kq}\Big[\underbrace{E^{(2)}dx - E^{(1)}dy}_{=0 \text{ by } \circledast}\Big] \Rightarrow C_x\,dx + C_y\,dy = 0$$

<u>Lets recap and clarify</u>: Let $\sigma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ be a soln of $\begin{matrix} \dot{x} = E^1(x) \\ \dot{y} = E^2(x) \end{matrix}$

We were given a fcn $C(x,y)$ and if $C(\sigma(t)) = \text{const}\ \forall t \in$ (some interval) then C is a 1^{st} Integral and it determines <u>the shape of the soln curves</u> (the level sets of C do). C does not give the full soln of time positions on the curves.

So we seek $\frac{d}{dt}C(\sigma(t)) \overset{!}{=} 0$ because RHS should be const $\frac{d}{dt}(\text{const}) = 0$

$$C_x\dot{x} + C_y\dot{y} \overset{!}{=} 0 \quad \text{or "mult by } dt\text{"} \Rightarrow C_x\,dx + C_y\,dy = 0$$

if $\langle x,y\rangle$ is on a curve $\sigma(t)$

and we showed

$$C_x\,dx + C_y\,dy = \frac{-y}{kq}\left[E^2dx - E^1dy\right] = 0 \checkmark$$

We could evaluate

$$DC_x\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = DC_x\begin{bmatrix}E^1\\ E^2\end{bmatrix} = C_xE^1 + C_yE^2 \overset{?}{=} 0$$

This would avoid any issues with $E^{(1)} = 0$

A&P ODE p.101

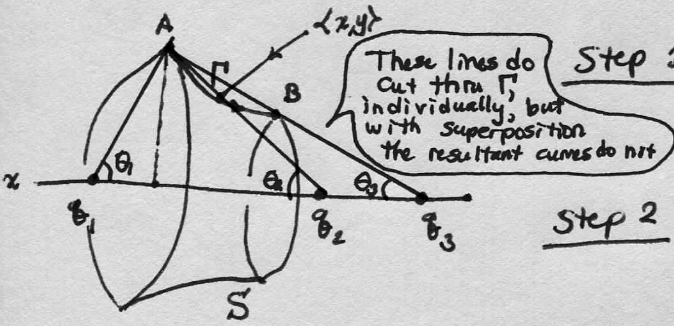
written up in Boyce & DiPrima Ch 2 Sheets

▷ There is another way to derive this, and it would work for n charges on a line. They can be both pos and neg

Let's just consider 3 charges. By the same sym arg, we can restrict to the pos xy plane

Smythe p.12
Jeans p. 57-58

consider a segment Γ of a flow line, not directly above a charge. Since flow lines don't cross, none cross Γ



Step 1 By sym, rotating Γ around x axis gives a surf S that is a section of a "tube" or deformed cylinder of flowlines with flat disc caps at A and B

Step 2 Consider the charge pts q_i individually. With no other charges, the lines of $\vec{E}$ would be radial. (of course, by superposition they end up forming the curves we seek)

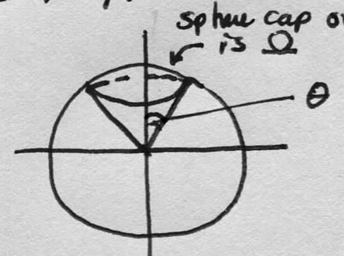
Draw lines from q_1, q_2, q_3 to A ~~and B~~ They make angles $\theta_1, \theta_2, \theta_3$ with the line of charges (x axis)

By $\boxed{\frac{1}{r^2}\text{-flux Cone Thm}}$ REA prob 19-50, M&T ch 7.4 writeup $\left\{\text{The lines of flux emanating from pt } q_1 \text{ and passing thru the disc cap at A}\right\} = \left\{\text{solid angle } \Omega_1 \text{ covered by disc on unit sphere}\right\}$

what is the solid angle of the disc at A in terms of θ_1?

From M&T ch 6 sheets 8a (not in M&T text)

solid angle $\Omega_1 = 2\pi(1-\cos\theta_1)$



Thus we see the flux due to charge q_1 is

$$q_1\Omega_1 = q_1 2\pi(1-\cos\theta_1)$$

Then the total flux from all charges is $q_1\Omega_1 + q_2\Omega_2 + q_3\Omega_3$

Step 3 Show the flux is the same thru any disc at a pt $\langle x,y \rangle \in \Gamma$ between A and B

There are no charges in the volume V bounded by S and the disc caps at A and B. By Gauss' Law, the flux thru disc B must equal flux thru disc A since none can pass thru sides S and it can't accumulate in the volume V. But the same logic applies if we picked any pt $\langle x,y \rangle$ on Γ between A and B and considered the associated disc.

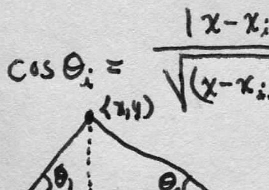
$\Rightarrow$ The flux is the same thru any disc, and the value $\tilde{\Phi}$ is the same for any $\langle x,y \rangle$ so long as it is on Γ

$$q_1\Omega_1 + q_2\Omega_2 + q_3\Omega_3 = \tilde{\Phi} \Rightarrow q_1 2\pi(1-\cos\theta_1) + q_2 2\pi(1-\cos\theta_2) + q_3 2\pi(1-\cos\theta_3) = \tilde{C}$$

$$(q_1+q_2+q_3)2\pi - 2\pi(q_1\cos\theta_1 + q_2\cos\theta_2 + q_3\cos\theta_3) = \tilde{\Phi}$$

$$\Rightarrow q_1\cos\theta_1 + q_2\cos\theta_2 + q_3\cos\theta_3 = \Phi$$ new const, depends on charges but not on pt $\langle x,y \rangle \in \Gamma$

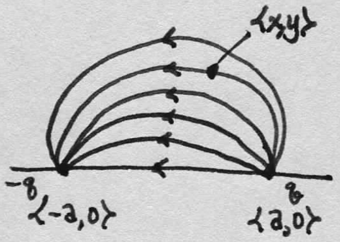
$$\cos\theta_i = \frac{|x-x_i|}{\sqrt{(x-x_i)^2+y^2}} \Rightarrow \underbrace{\frac{q_1|x-x_1|}{\sqrt{(x-x_1)^2+y^2}} + \frac{q_2|x-x_2|}{\sqrt{(x-x_2)^2+y^2}} + \frac{q_3|x-x_3|}{\sqrt{(x-x_3)^2+y^2}}}_{\text{This is the } 1^{st} \text{ Integral we were calling } C(x,y)} = \Phi$$



so really it should have contained $k q$ on sheet 10 $\longrightarrow$

so we end up with $C(x,y) = \dfrac{q_1\,|x-x_1|}{\sqrt{(x-x_1)^2+y^2}} + \dfrac{q_2\,|x-x_2|}{\sqrt{(x-x_2)^2+y^2}} + \dfrac{q_3\,|x-x_3|}{\sqrt{(x-x_3)^2+y^2}}$

Let's apply this to 2 opposite but equal charges that we considered before



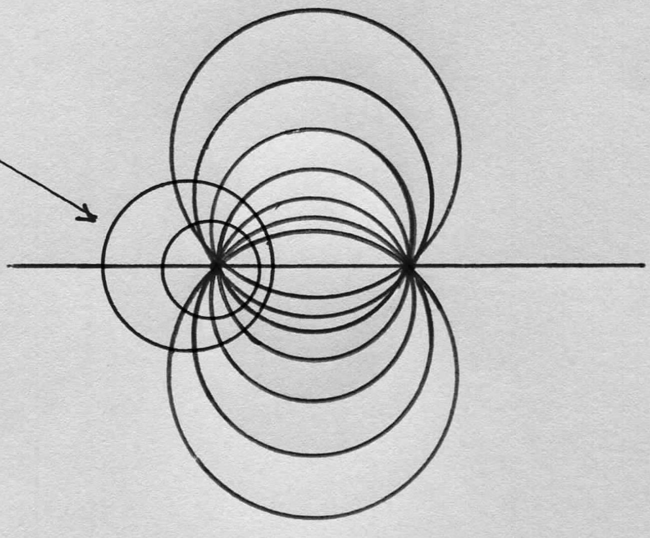
$$C = \frac{(-q)(x - -a)}{\sqrt{(x+a)^2+y^2}} + \frac{q\,(x-a)}{\sqrt{(x-a)^2+y^2}}$$

$$= \frac{-q\,(x+a)}{P^{1/2}} + \frac{q\,(x-a)}{N^{1/2}}$$

basically the same as before I should have kept the charges in on sheet 10 and factor $k = \frac{1}{4\pi\epsilon_0}$

Here is a curiosity: we get a remarkably similiar Picture just by drawing all circles thru 2 pts

H. Steinhaus Mathematical Snapshots p.143-144 He also draws circles of Apollonius that meet these at right angles are are const potential φ.



See also 'Tubes of Force' or 'Tubes of E flow' I wrote up in Ch 5 sheets