Feynman ch 4 6/1/2025

①

Summary and Extensions

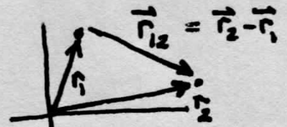
Maxwell's eqs: $\nabla\cdot E = \frac{1}{\epsilon_0}\rho$ $\qquad \nabla\cdot B = 0$

$\nabla\times E = -\frac{\partial}{\partial t}B \qquad \nabla\times B = \frac{1}{c^2}\left(\frac{\partial}{\partial t}\vec{E} + \frac{1}{\epsilon_0}\vec{J}\right)$

Let's start with the special case of electrostatics – all time derivs are 0

electro-statics $\begin{cases}\nabla\cdot E = \frac{1}{\epsilon_0}\rho \\ \nabla\times E = 0\end{cases}$ magneto-statics (ch 13) $\begin{cases}\nabla\cdot B = 0 \\ \nabla\times B = \frac{1}{\epsilon_0 c^2}\vec{J}\end{cases}$ They are decoupled – separate phenomena in this case.

<u>Ch 4.2</u> We are going to show the electrostatic eqs are equivalent to <u>Coulomb's Law</u> and <u>Principle of Superposition</u>



Given 2 position vectors $\vec{r}_1$ and $\vec{r}_2$, $\vec{r}_{12} := (\vec{r}_2 - \vec{r}_1)$ points from r_1 to r_2

Then for charges q_1 at r_1 and Q_2 at r_2:

$$\boxed{\vec{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 Q_2}{r_{12}^2}\hat{r}_{12} \quad \text{Coulomb's Law}}$$

This is valid for stationary charges. For moving charge(s) see next sheet →

This gives the force on charge q_1 due to charge Q_2. $\vec{F}_{12}$ may point towards or away from q_1 depending on the signs of the charges – key difference with gravity.

Restating this in terms of charges at points – force on pt x, with charge q_x there

$$\vec{F}(x) = \frac{1}{4\pi\epsilon_0}\frac{q_x Q_y}{\|y-x\|^3}(\vec{y}-\vec{x})$$

And for the Electric field $\boxed{\vec{E}(x) = \frac{1}{q_x}\vec{F}(x) = \frac{1}{4\pi\epsilon_0}\frac{Q_y}{\|y-x\|^3}(\vec{y}-\vec{x}) \quad \text{The force on a unit test charge at x due to charge } Q_y \text{ at y}}$

▷ Let's digress a moment to talk about units

Feynman uses mks units (SI units) $\frac{1}{4\pi\epsilon_0} \equiv \frac{c^2}{10^7} \approx 9.0\times10^9 \frac{N\cdot m^2}{C^2} \quad \frac{\text{newton}\cdot\text{meter}^2}{\text{Coulomb}^2} = \frac{\text{Volt}\cdot m}{C}$ (c: speed of light)

Permittivity of free space $\epsilon_0 = 8.85418782\times10^{-12}\ \frac{\text{farad}}{\text{meter}}$ LCL p. 43

<u>Purcell and M&T</u> use Gaussian cgs units $\qquad E_{gaus} = \sqrt{4\pi\epsilon_0}\,E \qquad q_{gaus} = \frac{1}{\sqrt{4\pi\epsilon_0}}q$

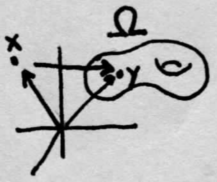
$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \rightsquigarrow E_g = \sqrt{4\pi\epsilon_0}\,E = \sqrt{4\pi\epsilon_0}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} = \sqrt{4\pi\epsilon_0}\frac{1}{4\pi\epsilon_0}\frac{(\sqrt{4\pi\epsilon_0}\,q_g)}{r^2}\hat{r} \Rightarrow E_g = \frac{q_g}{r^2}\hat{r}$$

and for Gauss' Law $\nabla\cdot E = \frac{1}{\epsilon_0}\rho \rightsquigarrow \sqrt{4\pi\epsilon_0}(\nabla\cdot E) = \sqrt{4\pi\epsilon_0}\frac{1}{\epsilon_0}\rho = \sqrt{4\pi\epsilon_0}\frac{1}{\epsilon_0}(\sqrt{4\pi\epsilon_0}\,\rho_g) \Rightarrow (\nabla\cdot E_g) = 4\pi\rho_g$

▷ If there is more than 1 charge, the total E field is just their sum <u>Superposition</u>

$$\vec{E}(x) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2(x)}\hat{r}_i(x)$$

Or for a continuum body of charge Ω



$$E(x) = \frac{1}{4\pi\epsilon_0}\int_{y\in\Omega}\frac{1}{\|y-x\|^3}(\vec{y}-\vec{x})\rho(y)\,d^3y = \frac{1}{4\pi\epsilon_0}\int_{\Omega}\frac{\rho(y)}{\|y-x\|^3}\vec{r}_x(y)\,d^3y$$

[we could write $\mathbb{R}^3$ in place of Ω since charge density $\rho\equiv0$ outside Ω]

Feynman uses tricks to avoid doing these kinds of integrals wherever possible. In some ways, this is a disservice. Lorrain, C, L just do it (cf their p.52)

▷ Aside: Electrostatic fields and moving charges

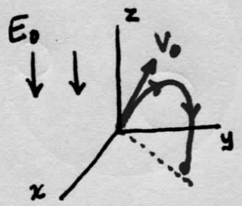
If charge Q is stationary and q is moving, then q feels the Coulomb's Law force from Q [on the other hand, Q would feel effects of velocity and accel of q and a time delay] cf LCL p. 43 and Griffiths p. 58

We can't have a charged particle q going into a stable elliptical orbit around Q, like a planet around the sun. It would emit radiation and the orbit would decay.

atoms manage to avoid this by quantum mechanics.

Classical Mech books like Fowles AM ch 4.5 give an example of a particle moving in an E field. We take the field to be unif and const (like gravity at the surf of Earth)

$$\vec{F} = q\vec{E} \qquad m\ddot{x} = qE \quad \Rightarrow \quad m\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = q\begin{bmatrix}E^1(x)\\ E^2(x)\\ E^3(x)\end{bmatrix} \qquad \text{let } \vec{E} = -E_0\hat{k}$$



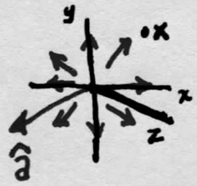
$$\ddot{x} = 0 \qquad \text{IC } \dot{\vec{x}}(0) = \vec{V}_0$$
$$\ddot{y} = 0$$
$$\ddot{z} = -\frac{q}{m}E \qquad \text{parabolic traj like a cannonball fired}$$

Ch 4.3 Electric Potential and Ch 4.4 $E = -\nabla\varphi$

Feynman gives a confusing discussion about $W := -\int E\cdot ds$ and "work done against the field". I am going to reformulate all this. But see my note on sheet 3

Given a fixed pos charge Q at the origin, what is the field E (force felt per unit charge) at pt x? $\vec{F} = q\vec{E}$

$\vec{x} = \vec{r}$ $r = \|x\|$



$$E(x) = \underbrace{\frac{Q}{4\pi\epsilon_0}}_{k}\frac{1}{\|x\|^3}\vec{x} = \frac{+k}{r^2}\hat{e}_r \quad \text{spherical sym}$$

ElectroStatics has $\nabla\times E = 0$ so we can define a potential fcn φ that will generate $\vec{E}$ and $\int_\sigma E\cdot ds$ depends only on endpts of σ.

[For now, let's skip all physics conventions about neg signs, and just do the math following M&T ch 7.3 sheets and Fowles AM ch 6 sheets (gravity)]

Let $\hat{a}$ be arb unit direction vector. Reference pt $x_R = R\hat{a}$, $x = r\hat{a}$ for pos numbers $r < R$

Since we have path independence, use straight line $\sigma(t) = (1-t)R\hat{a} + tr\hat{a}$ $0 \le t \le 1$

$\sigma(0) = R\hat{a} = x_R$

$\|\sigma\| = (1-t)R + tr > 0$ $\quad \sigma(1) = r\hat{a}$

$\|a\| = 1$ $\quad \sigma'(t) = (r-R)\hat{a}$ $(r-R)$ is neg

Define potential fcn (vf generating fcn)

$$\varphi_{R\hat{a}}(x) = \int_\sigma \vec{E}\cdot d\vec{s} = \int_{x_R}^{x}\vec{E}\cdot d\vec{s} = \int_0^1 \frac{k}{\|\sigma\|^3}\sigma\cdot\sigma'\,dt$$

$$= k\int_0^1 \frac{1}{[tr+(1-t)R]^{\cancel{3}\,2}}\cancel{(tr+(1-t)R)\hat{a}\cdot\hat{a}}\,(r-R)\,dt$$

$$= k\int_0^1 \frac{1}{[tr+(1-t)R]^2}(r-R)\,dt$$

COV
$u = tr + (1-t)R$
$du = (r-R)dt$

$$= k\int_{u=R}^{r}\frac{1}{u^2}\,du = -k\left[\frac{1}{r} - \frac{1}{R}\right]$$

(Griffiths p. 85 emphasizes the formulas depend on this)

$\boxed{\text{But we want the ref pt to be } \infty}$, so let $R \to \infty$ $\quad \varphi_\infty(x) = -\frac{Q}{4\pi\epsilon_0}\frac{1}{\|x\|}$

just omit this going forward

Observe $\nabla\varphi = \nabla\left(\frac{-k}{r}\right) = -k\nabla\left(\frac{1}{r}\right)$ $\leftarrow$ M&T ch 3 $\nabla(r^n) = nr^{n-2}\vec{r}$ Done here sheet 4

$= -k\left(\frac{-1}{r^3}\vec{r}\right) = \frac{k}{r^3}\vec{r} = E$ ✓ See sheet 4

This does mean for a pos charge Q, the potential is neg, which Griffiths p. 84 doesn't like.

So it looks good, but to match up with what the physics books have, we must introduce the convention $E \overset{!}{=} -\nabla\varphi$ (force points toward decreasing potential)

Cont'd →

we know $E(x) = \dfrac{+k}{\|x\|^3}\vec{x}$ so we must re-define $\boxed{\varphi(x) := \int_\infty^x (-E)\cdot ds}$

Then $\nabla\varphi = -E$ and $\boxed{\varphi(x) = \dfrac{+Q}{4\pi\epsilon_0}\dfrac{1}{\|x\|} = \dfrac{Q}{4\pi\epsilon_0}\dfrac{1}{r}}$

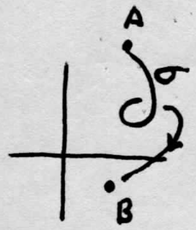
▷ we know work $W = \Delta PE$ (no KE since charges are stationary at beginning and end) electrostatics

$$= \varphi_{final} - \varphi_{initial}$$

$$= \varphi(x) - \underbrace{\varphi(\infty)}_{0}$$

$$= \varphi(x) = \int_\infty^x -E\cdot ds = -\int_\infty^x E\cdot ds$$

That is how there is a minus sign on the expression for work



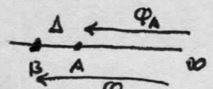
$F = qE$

Work (per unit charge) to transport q from A to B

$$\frac{W_{AB}}{q} = -\int_\sigma E\cdot ds = \int_A^B \nabla\varphi\cdot ds = \varphi(B) - \varphi(A) \quad \text{Potential Difference}$$

[Voltage]

Note also



$$\int_\infty^B - \int_\infty^A = \int_A^B$$

$\varphi_A + \Delta = \varphi_B$

$\Delta = \varphi_B - \varphi_A$

Given a charge Q at origin, the work done bringing a charge q in from ∞ to a position with radial dist r is

$$\frac{W}{q} = -\int_\infty^r E\cdot ds = \varphi(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r} \Rightarrow \boxed{W = \frac{1}{4\pi\epsilon_0}\frac{qQ}{r}}$$

Remark: Work being pos along some path γ does not mean a free particle will flow along γ!

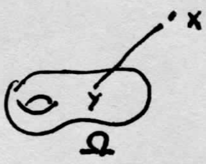
Note that bringing in pos charge q towards pos Q is doing work against the field E.

▷ φ also obeys superposition principle, since E does.

$$E = E_1 + E_2$$

$$-\nabla\varphi = -\nabla\varphi_1 + -\nabla\varphi_2 \Rightarrow -\nabla(\varphi = \varphi_1 + \varphi_2)$$

$$\varphi(x) = \sum_{i=1}^{N} \frac{k\, q_i}{\|x - y_i\|}$$



$$\varphi(x) = \frac{1}{4\pi\epsilon_0}\int_{y\in\Omega} \frac{\rho(y)\, d^3y}{\|x - y\|}$$

Let's illustrate this with practical calculations ⟶

5/26/2025

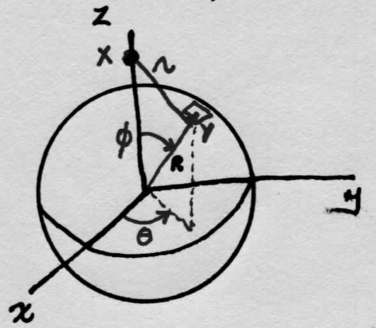
Griffiths p.85

3a

Let's illustrate $\varphi(x) = \frac{1}{4\pi\epsilon_0}\int_{y\in\Omega} \frac{\rho(y)\, d^3y}{\|y-x\|}$ in some situations we need later.

$K := \frac{1}{4\pi\epsilon_0}$

Let $S := S^2_{(0,R)}$ sphere (shell) which carries a unif charge Q



$$\varphi(x) = \int_S \varphi(x;y)\, dy$$

I am writing $\varphi(x;y) = \frac{K q}{\|y-x\|}$ Could even put $\varphi_{\infty}(x;y)$

$$= \int_S \frac{K\rho(y)\, d^3y}{\|y-x\|}$$

$$= K\int_S \frac{\sigma}{r(y)}\, dA$$

S is 2-dim and charge distrib is unif so $\sigma = \frac{Q}{4\pi R^2}$ const

How can we express $r(y) = \|y-x\|$? We choose co-ord axes so $x = \langle 0,0,z\rangle$

By law of Cos $c^2 = a^2+b^2-2ab\cos\gamma$ so $r^2 = R^2+z^2-2Rz\cos\phi$

$$\varphi(x) = K\sigma\int_{\theta=0}^{2\pi}\int_0^{\pi} \frac{1}{\sqrt{R^2+z^2-2Rz\cos\phi}} R^2\sin\phi\, d\phi\, d\theta = 2\pi R^2 K\sigma\int_0^{\pi}\frac{1}{\sqrt{R^2+z^2-2Rz\cos\phi}}\sin\phi\, d\phi$$

$u = R^2+z^2-2Rz\cos\phi$

$du = 2Rz\sin\phi\, d\phi$

$$= \frac{\cancel{2}\pi R^{\cancel{2}} K\sigma}{\cancel{2}\cancel{R}z}\int_{\phi=0}^{\pi} u^{-1/2}\, du = \frac{\pi R K\sigma}{z}\left[\frac{1}{1/2}u^{1/2}\right]_{\phi=0}^{\pi}$$

$$= \frac{2\pi R K\sigma}{z}\left[\sqrt{R^2+z^2-2Rz\underbrace{\cos(\pi)}_{-1}} - \sqrt{R^2+z^2-2Rz\underbrace{\cos(0)}_{1}}\right]$$

$R^2+z^2+2Rz \qquad R^2+z^2-2Rz$

$(R+z)^2 \qquad (R-z)^2$

But we must take pos root so $R-Z$ if Z inside, $Z-R$ if Z outside

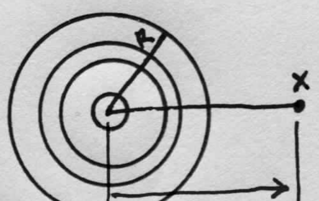
$$= \frac{\cancel{2\pi}\cancel{R}K}{z}\frac{Q}{\cancel{2}\cancel{4\pi}R^{\cancel{2}}}\begin{bmatrix} R+z-(R-z) = 2z & \text{inside} \\ R+z-(z-R) = 2R & \text{outside}\end{bmatrix}$$

$$\Rightarrow \boxed{\varphi(x) = \begin{cases} \frac{Q}{\cancel{z}4\pi\epsilon_0 R}\cancel{z} & \text{inside} \longrightarrow \frac{1}{4\pi\epsilon_0}\frac{Q}{R} \text{ const} \\ \frac{Q}{z4\pi\epsilon_0}\frac{\cancel{R}}{\cancel{R}} & \text{outside} \longrightarrow \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \; \triangle \end{cases}}$$

($z = \|x\| = r$ since Z axis was arb choice)

Observe we can let $r \searrow R$ and get continuity $\varphi(R) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R}$ on surf even though this would have been a singularity.

$\triangle$ Now consider a solid ball $B(0,R)$ — redefine density to vol density ρ



$$\varphi_B(x) = \int_{\text{all } S_R} \varphi_{S_R}(x) = \int \frac{K}{r}\, dQ = \frac{K}{r}\int_{R=0}^{R} \rho(4\pi R^2)\, dR = \frac{K}{r}\rho\left(\frac{4}{3}\pi R^3\right)$$

const

vol $(B(0,R))$

$$\boxed{= \frac{1}{4\pi\epsilon_0}\frac{Q}{r}}$$ outside

just like pt charge for x outside as expected

But what about potential <u>inside</u>? More complicated. See → sheet 8

For reference, here are some calculations from $\boxed{\text{M\&T ch 3.5 p.196}}$

8 $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad r = \|\vec{r}\| = (x^2+y^2+z^2)^{1/2} = p^{1/2}$

a Show $\boxed{\nabla(r^n) = n\, r^{n-2}\, \vec{r}}$

$\nabla(r^n) = \nabla(p^{n/2})$ and $\frac{\partial}{\partial x}(p^{n/2}) = \frac{n}{2} p^{\frac{n}{2}-1} \frac{\partial p}{\partial x} = \frac{n}{\cancel{2}} p^{\frac{n-2}{2}} (\cancel{2}x)$

$= n\,(p^{1/2})^{n-2} x = n\, r^{n-2} x$

From this pattern, we can see $\nabla(r^n) = \begin{bmatrix} n r^{n-2} x \\ n r^{n-2} y \\ n r^{n-2} z \end{bmatrix} = n r^{n-2} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = n r^{n-2} \vec{r}$ □

c Show $\boxed{\nabla\cdot(r^n \vec{r}) = (n+3) r^n}$

$$\nabla\cdot(r^n \vec{r}) = \nabla\cdot\left(p^{n/2} \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \frac{\partial}{\partial x}\left(p^{n/2} x\right) + \frac{\partial}{\partial y}\left(p^{n/2} y\right) + \frac{\partial}{\partial z}\left(p^{n/2} z\right)$$

$$= \left(\frac{n}{\cancel{2}} p^{\frac{n}{2}-1} \cancel{2}x \cdot x + p^{n/2}\right) + \left(n (p^{1/2})^{n-2} y^2 + p^{n/2}\right) + \left(n\,(p^{1/2})^{n-2} z^2 + p^{n/2}\right)$$

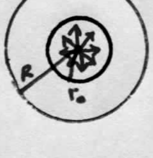
$$= n\, r^{n-2}(x^2+y^2+z^2) + 3\,(p^{1/2})^n = n r^{n-2} r^2 + 3 r^n = (n+3)\, r^n$$

This would be 0 if $n=-3$ and only this value of n

Thus $\nabla\cdot(r^{-3}\vec{r}) = \nabla\cdot\left(\frac{1}{r^3}\vec{r}\right) = 0$ This is really inverse squared: $\nabla\cdot\left(\frac{1}{r^2}\overset{\hat{r}}{\hat{e}_r}\right) = 0$

ch 4-5 and 4-6 E Flux and Gauss' Law

▷ Before considering electric field E flux from a charge, lets consider flux from light rays eminating from a pt source. Light is a stream of photons, energy is conserved.



steady state - any conserved quantity (like energy) flowing thru small sphere $S(0,r_0)$ per sec must also flow thru big sphere $S(0,R)$ per sec. Area sphere $4\pi R^2$ Let $R = \lambda r_0$

Let $h(r)$ be energy per unit area. What is the form of h?

$$\mathcal{E} = h(r_0)\,\cancel{4\pi r_0^2} \overset{!}{=} h(R)\, 4\pi R^2 = h(\lambda r_0)\,\cancel{4\pi}\,\lambda^2 \cancel{r_0^2} \Rightarrow h(\lambda) = \frac{1}{\lambda^2} h(r_0)$$

So this shows how a conserved quantity varies as $\frac{1}{r^2}$ as it spreads out.

BUT this does NOT apply to E!

this does NOT apply to E! BUT this does NOT apply to the E field.

No conserved quantity is radiating out from a charge, making the E field.

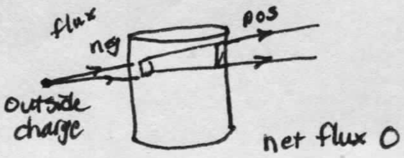
(the charge doesn't lose anything as it makes E field)

So it is just by experiment that we see Coulomb's Law has $\frac{1}{r^2}$ dependance.

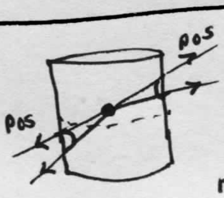
Gauss' Law

$$\int_S \vec{E}\cdot\hat{n}\, dA = \frac{1}{\epsilon_0} Q$$

flux thru surf S — charge enclosed by surf



net flux 0



inside charge non-zero flux

Feynman ch 4 4/20/2025 Easter 5

Let's give the Gauss' Law discussion from M&T ch 7.4 p.447

Here is where $1/r^2$ dependence becomes critical. $\nabla\times E = 0$ true for any central vf

Thm 10 (Gauss' Law) Let M be a smooth 3-mfd in $\mathbb{R}^3$ (cpt "solid body")
vf $\vec{E} = \frac{1}{r^2}\hat{e}_r = \frac{1}{r^3}\vec{r}$ inv sq vf with 1 singularity at origin
origin $\langle 0,0,0\rangle \notin \partial M = S$ surf of M

$$\Rightarrow \underbrace{\int_{\partial M} \frac{1}{r^2}\hat{e}_r\cdot n\,dA}_{\text{flux of } E} = \begin{cases} 4\pi & \text{origin in } M \\ 0 & \text{origin outside } M \end{cases}$$

Pf **Case 1** origin outside M so no singularities in domain

$$\int_{\partial M}\frac{1}{r^3}\vec{r}\cdot\hat{n}\,dA \overset{\text{Div Thm}}{=} \int_M \nabla\cdot\left(\frac{1}{r^3}\vec{r}\right)dV = \int_M 0\,dV = 0$$

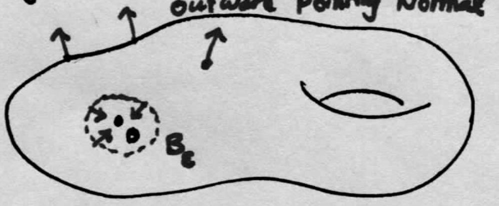
See calculation on sheet 4 $\nabla\cdot\left(\frac{1}{r^3}\vec{r}\right) = 0$

Case 2 origin inside M (singularity of vf)

we can't apply Div Thm to M since there is a singularity, but we can excise it by cutting out a small open ball around it. No singularities, apply Div Thm:

Define new 3-mfd $N := M \setminus \mathring{B}(0,\varepsilon)$

$$\partial N = \partial M \cup \partial B_\varepsilon$$



$$\int_{\partial N}\frac{1}{r^3}\vec{r}\cdot\hat{n}\,dA \overset{\text{Div}}{=} \int_N \nabla\cdot\left(\frac{1}{r^3}\vec{r}\right)dV = 0$$

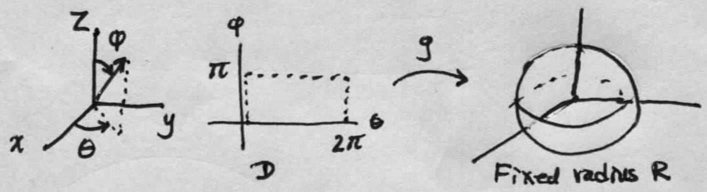
$$\| $$

$$\int_{\partial M}\frac{1}{r^3}\vec{r}\cdot\hat{n}\,dA + \int_{\partial B_\varepsilon}\frac{1}{r^3}\vec{r}\cdot\hat{n}\,dA \implies \int_{\partial M}\frac{1}{r^3}\vec{r}\cdot n\,dA = -\int_{\partial B_\varepsilon}\frac{1}{r^3}\vec{r}\cdot\hat{n}\,dA \quad ★$$

remember this

As part of ∂N, this has the opposite $\mathcal{O}^r$ as usual — it has Inward Pointing Normal

M&T ch 6.6 sheet 14 does this calculation:



$$\int_{S=g(D)}\vec{F}\cdot n\,dA = \int_D \vec{F}(g(\theta,\varphi))\cdot(\vec{g}_\theta\times\vec{g}_\varphi)\,d\theta\,d\varphi$$

$$g:(\theta,\varphi)\longmapsto R\begin{bmatrix} C_\theta S_\varphi \\ S_\theta S_\varphi \\ C_\varphi\end{bmatrix} = R\,\hat{e}_r$$

This happens to also be an expression for Outward normal.

$$Dg_{\theta\varphi} = R\begin{bmatrix} -S_\theta S_\varphi & C_\theta C_\varphi \\ C_\theta S_\varphi & S_\theta C_\varphi \\ 0 & -S_\varphi \end{bmatrix} \qquad \vec{g}_\theta\times\vec{g}_\varphi = -R^2\begin{bmatrix} C_\theta S_\varphi^2 \\ S_\theta S_\varphi^2 \\ S_\varphi C_\varphi\end{bmatrix}$$

$$\begin{array}{cc} \vec{g}_\theta & \vec{g}_\varphi \end{array}$$

$$= R^2 S_\varphi\begin{bmatrix} -C_\theta S_\varphi \\ -S_\theta S_\varphi \\ -C_\varphi\end{bmatrix} = -R^2 S_\varphi\,\hat{e}_r$$

Inward pointing normal

cont'd →

Thus we can compute

$$\int_{\partial B_\varepsilon} \frac{1}{r^3}\vec{r}\cdot\hat{n}\,dA = \int_{S^2_{(0,\varepsilon)}} \frac{1}{\cancel{\varepsilon^2}}\hat{e}_r\cdot\left(-\cancel{\varepsilon^2}\sin\varphi\,\hat{e}_r\right)d\theta\, d\varphi = -\int_{\theta=0}^{2\pi}\int_0^{\pi}\sin\varphi\, d\theta\, d\varphi = -2\pi\int_0^{\pi}\sin\varphi\, d\varphi$$

$$= +2\pi\underset{-1\;\;-1}{\Big[\cos\varphi\Big]_0^{\pi}} = -4\pi$$

But ★ : $\displaystyle\int_{\partial M}\frac{1}{r^3}\vec{r}\cdot n\,dA = -\int_{\partial B_\varepsilon}\frac{1}{r^3}\vec{r}\cdot\hat{n}\,dA = -\left[-4\pi\right] = 4\pi$ $\boxed{\text{QED}}$

▷ How do we get electric charge into this? Mult by $\frac{q}{4\pi\epsilon_0}$ to get Coulomb vf E:

$$\frac{q}{4\pi\epsilon_0}\int_{\partial M}\frac{1}{r^3}\vec{r}\cdot\hat{n}\,dA = \frac{q}{\cancel{4\pi}\epsilon_0}(\cancel{4\pi}) \Rightarrow \boxed{\int_{\partial M}\frac{q}{4\pi\epsilon_0 r^2}\hat{e}_r\cdot\hat{n}\,dA = \frac{1}{\epsilon_0}q}$$

▷ For a single charge at pt x_0, we'd just have

$$\frac{1}{4\pi\epsilon_0}\int_{\partial M}\frac{q}{\|x-x_0\|^3}(\overrightarrow{x-x_0})\cdot\hat{n}\,dA = \frac{1}{\epsilon_0}q$$

▷ For N charges: (charges inside)

$$\int_{\partial M}\underbrace{\left(\sum_{i=1}^{N}\frac{1}{4\pi\epsilon_0}\frac{q_i\,(\overrightarrow{x-x_i})}{\|x-x_i\|^3}\right)}_{\vec{E}}\cdot\hat{n}\,dA_x \overset{\text{interchange}}{=} \sum_{i=1}^{N}\frac{q_i}{4\pi\epsilon_0}\underbrace{\left[\int_{\partial M}\frac{1}{\|x-x_i\|^3}(\overrightarrow{x-x_i})\cdot\hat{n}\,dA\right]}_{4\pi} = \sum\frac{q_i\,\cancel{4\pi}}{\cancel{4\pi}\epsilon_0}$$

$$= \frac{1}{\epsilon_0}\underbrace{\sum^{N} q_i}_{Q}$$ total charge inside M

$$\boxed{=\frac{1}{\epsilon_0}Q}$$

▷ For a continuum body of charge Ω inside M

$$\int_{\partial M}\underbrace{\left[\int_{\Omega}\frac{\overset{\rho(y)d^3y}{dq}}{4\pi\epsilon_0}\frac{1}{\|x-y\|^3}(\overrightarrow{x-y})\right]}_{\vec{E}}\cdot n\,dA \overset{\text{interchange}}{=} \int_{\Omega}\frac{\rho(y)d^3y}{4\pi\epsilon_0}\left[\underbrace{\int_{\partial M}\frac{1}{\|x-y\|^3}(\vec{x}-y)\cdot n\,dA}_{4\pi}\right]$$

$$= \frac{1}{\epsilon_0}\int_{\Omega}\rho(y)\,d^3y = \boxed{\frac{1}{\epsilon_0}Q \text{ total charge inside } \Omega\subset M}$$

We can also write:

$$\int_{\partial M}\vec{E}\cdot\hat{n}\,dA = \frac{1}{\epsilon_0}\int_M\rho(y)\,d^3y$$ since $\rho\equiv 0$ outside Ω and $\Omega\subset M$

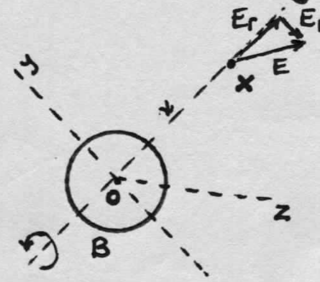
Div Thm $\rightarrow$ ||

$$\int_M(\nabla\cdot E)\,dV = \int_M\frac{\rho}{\epsilon_0}dV \Rightarrow \int_M\left(\nabla\cdot E-\frac{\rho}{\epsilon_0}\right)dV = 0 \Rightarrow \boxed{\nabla\cdot E = \frac{1}{\epsilon_0}\rho}$$

Since this must hold for arb Ω, M $\Omega\subset M$

Feynman ch 4.7 and M&T p.398 (ch6.6) and p.450 #15 (ch7.4)

By spherical rotation sym, we can determine the $\vec{E}$ field for a pt charge, or unif sphere of charge. Show the field must be $\boxed{\vec{E} = E(r)\hat{e}_r}$ [Sym Argument]



choose arb pt x outside B (could be a pt, spherical shell, or solid)

Suppose $E(x) = E_r + E_p$ (E_p is perp component, non radial)

Imagine a co-ord sys with origin at center of B and x axis goes radially thru pt x

If I imagine rotating co-ord axes y, z around x axis, component E_p would change.

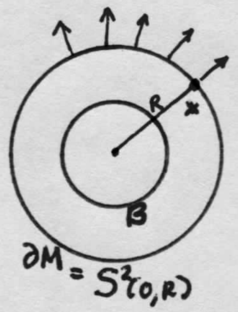
BUT B looks identical after the imaginary change of co-ords, so E_p should look like it is in its original position $\Rightarrow\!\Leftarrow$

E_p must be 0 and $\vec{E}$ can only depend on r, not θ, φ angles.

▷ Given this Sym, we can use Gauss' Law to derive Coulomb's Law.

[Gauss' Law is said to also apply to moving charges, so more general than Coulomb's, but we don't prove that here.]

cf LCL p.51



Let B have charge Q. [a field that is $1/r^2$ but does depend on angles θ, φ can still satisfy Gauss Purcel EM p.24]

Image a larger sphere, concentric with B and going thru x : $S^2(0,R)$

From Gauss, $\displaystyle\int_{\partial M} \vec{E}\cdot\hat{n}\,dA = \frac{1}{\epsilon_0}Q$

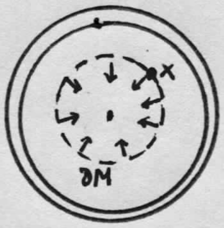
$$\int_{S^2(0,R)} E(R)\,\overbrace{\cancel{\hat{e}_r}\cdot\cancel{\hat{e}_r}}^{1}\,R^2\sin\varphi\,d\theta\,d\varphi \Rightarrow E(R)\,4\pi R^2 = \frac{1}{\epsilon_0}Q$$

$$\Rightarrow \underset{E(x)}{E(R)} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}$$

Then if we had a charge q at x, $\vec{F} = q\vec{E}$

Force on charge at x $\quad \vec{F}(x) = \dfrac{1}{4\pi\epsilon_0}\dfrac{Qq}{R^2}\hat{e}_r$ and we recover Coulomb's law

▷ What if there was a hollow shell (concentric spherical cavity inside)? Show $\vec{E} = 0$ inside, just like for Gravity. [Things become more complex when B can be a metal conductor, coming up]



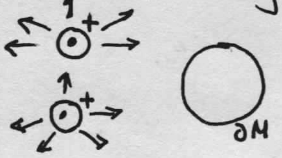
Again from sym, we know $\vec{E} = E(r)\hat{e}_r$ so on smaller concentric sphere $S^2(0,\lambda) = \partial M$, it has a const magnitude in all radial directions. What is this const magnitude

$\displaystyle\int_{\partial M} \vec{E}\cdot n\,dA = 0$ (no charge inside)

$\displaystyle\int_{S(0,\lambda)} E(\lambda)\,e_r\cdot e_r\,dA = 0 \quad \Rightarrow \quad E(\lambda)\,4\pi\lambda^2 = 0$

$\Rightarrow \quad E(\lambda) = 0$ for any x inside cavity □

Note this arg fails for a sphere ∂M with charges outside but not concentric



again no charge inside, $\displaystyle\int_{\partial M} E\cdot n\,dA = 0$

But no sym, so we can't solve for E, we don't know anything about its mag and direction at all pts of ∂M