Eves ASOG
ch 2

1/31/2025

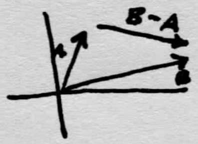
1

p.54

We can assign a dir (θ^n) on a line or line seg.
Eves writes $\overline{AB}$ but I will write $\overrightarrow{AB}$ $(A \to B)$ Not quite a vector.

$\overline{AB} = -\overline{BA}$

we can think of it mnemonically as vectors in $\mathbb{R}^2$ (or on $\mathbb{R}^1$) but Eves is only working on a line.



For signed dist (signed magnitudes)
$AB = b - a$ as a number in $\mathbb{R}$

p.55

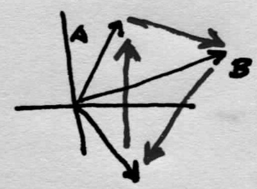
Def Range of pts - pts on a line - think discrete finite set - not Cantor set, limit pts or any of that

Base - it is the line of the "range" - all the pts lie on this line

Call the pts - continuum on the line would be "complete range".

Thm 2.1.3 A, B, C any 3 pts on a line $\Rightarrow \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$
cylic segment sum
$(b-a) + (c-b) + (a-c) = 0$

This is easily seen as vectors

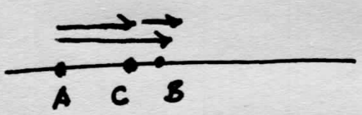


we could then imagine these vectors on $\mathbb{R}$ axis

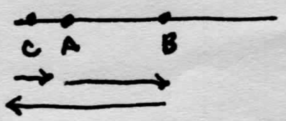
pf. A can be arb

Case 1


$AB + BC = AC \rightarrow AB + BC \underset{+CA}{-AC} = 0$

Case 2


$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} \Rightarrow AB \underset{+CA}{-AC} \underset{+BC}{-CB} = 0$

Case 3


$CA + AB = CB \Rightarrow CA + AB$

For any 3 pts on line A, B, O, form
$AB + BO + OA = 0$ $\boxed{\text{Thm 2.1.3}}$
$AB = -BO - OA$
$= OB - OA$ □

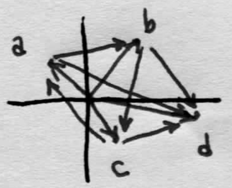
Thm 2.1.4 $O \in$ ~~segment~~ line $\overleftrightarrow{AB} \Rightarrow AB = OB - OA$
pf

Thm 2.1.5 (Euler) A, B, C, D on a line $\Rightarrow \overrightarrow{AD} \cdot \overrightarrow{BC} + \overrightarrow{BD} \cdot \overrightarrow{CA} + \overrightarrow{CD} \cdot \overrightarrow{AB} = 0$

What does this mean? what is this operation? (Eves means it as ordinary mult in $\mathbb{R}$)

If we regard pts in $\mathbb{R}^2$ vectors

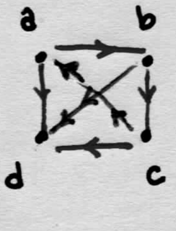
dot prod of vectors - reduces to ord mult for all vectors only in $\mathbb{R}$



$(\vec{d} - \vec{a}) \cdot (c - b) + (d - b) \cdot (a - c) + (d - c)(b - a)$

$d \cdot c - d \cdot b - a \cdot c + a \cdot b$
$+ d \cdot a - d \cdot c - b \cdot a + b \cdot c$
$+ d \cdot b - d \cdot a - c \cdot b + c \cdot a = \vec{0}$

Then we can restrict vectors to $\mathbb{R}$



How does Eves do it? use $\boxed{2.1.4}$
$BC = DC - DB$
$CA = DA - DC$
$AB = DB - DA$

Now just expand $AD(DC - DB) + BD(DA - DC) + CD(DB - DA)$

□

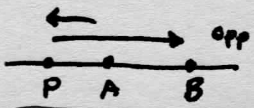
Eves p. 56

Let A, B, P be distinct pts on a line. (Doesn't matter which way we σ^t line)

Def "ratio in which P divides $\overline{AB}$" $:= \dfrac{\overrightarrow{AP}}{\overrightarrow{PB}}$ Just accept these words as meaning this quantity!

Define Real $r := \dfrac{\overrightarrow{AP}}{\overrightarrow{PB}}$ (if $P=A$, $r=0$ if $P=B$, $r=\infty$)

wlog take $|AB| = 1$



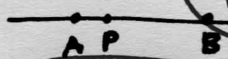
neg for P outside

$-1 < r < 0$

$r = \dfrac{AP}{PB} = \dfrac{-p}{p+1}$ p = dist from A

$= 0$ if $p = A$

l'hop $= \dfrac{-1}{1} = -1$ $p \to \infty$

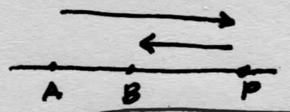


Pos for P inside

$0 < r < \infty$

$\dfrac{AP}{PB} = 0$ $P = A$

$\dfrac{AP}{PB} = \infty$ $P \to B$



p = dist from B

neg for P outside

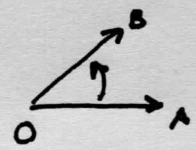
$-\infty < r < -1$

$\dfrac{AP}{PB} = \dfrac{p+1}{-p}$

$p = 0 \Rightarrow -\infty$

$p = \infty \Rightarrow -1$

Def $\measuredangle_{\overrightarrow{AOB}}$



rotate $\overline{OA}$ until it coincides with $\overline{OB}$

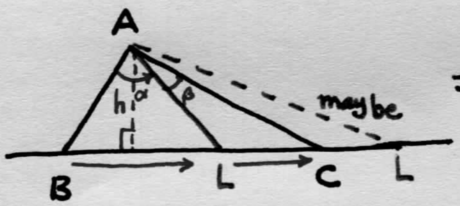
CCW is pos

angle must be less than 180°, so the sign is pos or neg depending on which way accomplishes this.

σ^t of Δ:

Eves writes $\Delta_{\overrightarrow{ABC}}$

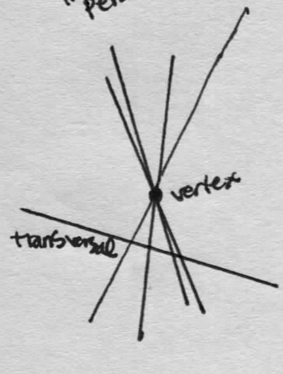
Thm 2.1.9



$\Rightarrow$ Signed distances $\dfrac{\overrightarrow{BL}}{\overrightarrow{LC}} = \dfrac{AB \sin \measuredangle_{BAL}}{AC \sin \measuredangle_{LAC}} = \dfrac{AB \sin\alpha}{AC \sin\beta}$

"ratio L divides $\overline{BC}$"

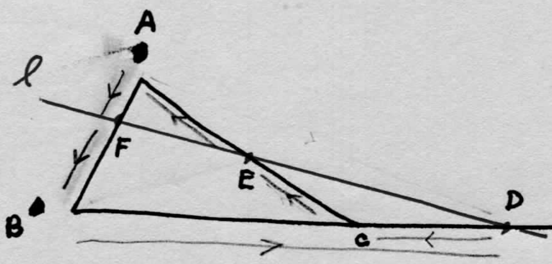
"pencil" of lines



Eves p. 64
ch 2

1/5/2025

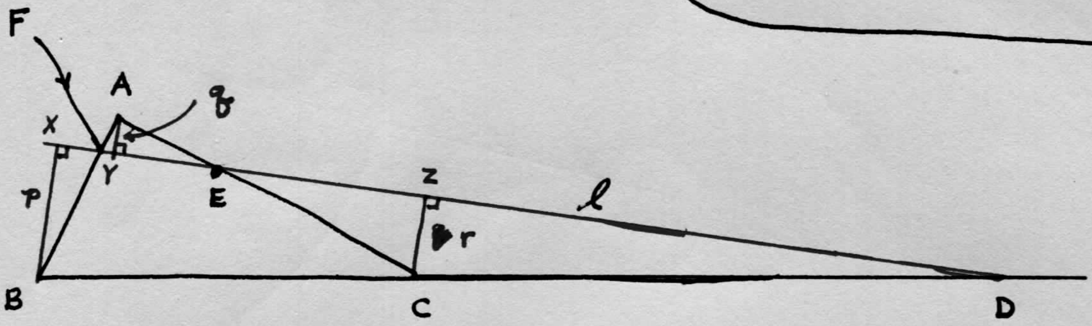
③



Menelaus' Line Thm

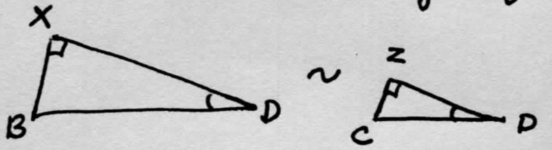
Given Δ_{ABC}, choose pts F, E on each side, and extend the base and choose pt D

(Pts D, E, F line on a line (call it ℓ)) $\Leftrightarrow$ $\left(\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = -1 \right)$



There is a uniq $\perp$ line seg to B from a pt we shall name X, length p
Likewise A to pt Y, length q and C [redacted] connects to ℓ at pt Z, length r

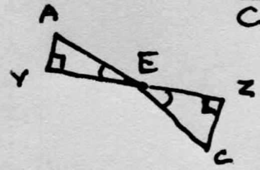
(1)



By AA similiarity
Thus hypot $|BD| = \lambda |CD|$
$|BX| = \lambda |CZ|$

i.e. $\frac{BD}{CD} = \lambda = \frac{BX}{CZ} = \frac{p}{r}$

(2) These triangs are similiar

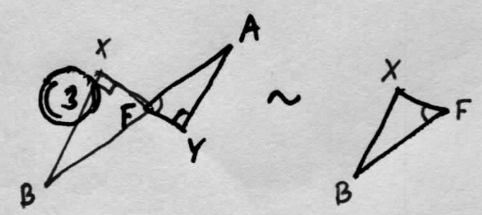


vertex-ical angles at E — AA similiarity

$|CE| = \lambda |AE|$
$|CZ| = \lambda |AY|$

$\frac{CE}{AE} = \lambda = \frac{CZ}{AY} = \frac{r}{q}$

(3)



Vertex-ical angles at F
$|BF| = \lambda |FA|$
$|BX| = \lambda |AY|$ $\Rightarrow \frac{|FA|}{|BF|} = \lambda = \frac{|AY|}{|BX|} = \frac{q}{p}$

Then we want $\frac{\overrightarrow{BD}}{\overrightarrow{DC}} \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \frac{\overrightarrow{AF}}{FB}$ but what we've got is $\frac{|BD|}{|CD|} \frac{|CE|}{|AE|} \frac{|FA|}{|BF|}$

$= \frac{\cancel{p}}{\cancel{r}} \frac{r}{q} \frac{\cancel{q}}{\cancel{p}} = 1$

and we can transpose some internal letters to say the same thing: $\frac{|BD|}{|DC|} \frac{|CE|}{|EA|} \frac{|AF|}{|FB|} = 1$

But now we are consistently moving along the segment from one pt to another so if we remove the abs val signs

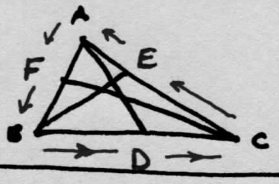
$\frac{\overrightarrow{AF}}{\overrightarrow{FB}}$ denotes going from A to middle pt F and then F to B — one consistent direction so this is pos

Likewise $\frac{\overrightarrow{CE}}{\overrightarrow{EA}}$ is pos, but $\frac{BD}{DC}$ means doubling back since D is not in the interval $\Rightarrow$ neg sign $\Rightarrow \frac{\overrightarrow{BD}}{\overrightarrow{DC}} \frac{\overrightarrow{CE}}{\overrightarrow{EA}} \frac{\overrightarrow{AF}}{\overrightarrow{FB}} = -1$ only because we had to extend base of triang, and D outside

□

Eves p. 65-66 1/12/2025

Ceva Pt Thm



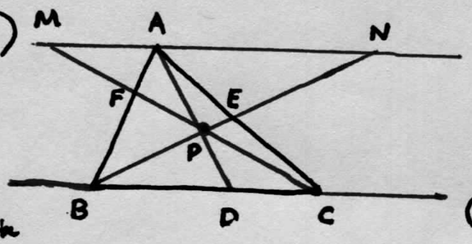
Draw line segments from each vertex to some pt on opp side (not vertex) "cevian"

$\exists$ Routh's $\frac{1}{7}$ Thm

The line segments meet at a pt P $\iff$ $\dfrac{\overrightarrow{BD}}{\overrightarrow{DC}}\,\dfrac{\overrightarrow{CE}}{\overrightarrow{EA}}\,\dfrac{\overrightarrow{AF}}{\overrightarrow{FB}} = +1$ signed lengths, going around Δ CCW

pf

($\Rightarrow$)

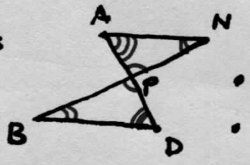


This is a different pf than in my Jacobs Euclidean Geo sheet (41)

cf youtube Michael Greenberg

① Draw line thru A parallel to base $\overline{BC}$
Extend CF to M and BE to N

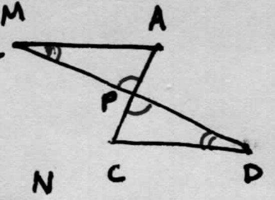
② Similar triangs



all 3 angles match:
- vertex-ical angles at P
- B & N parallel lines cut by transv
- D & A same arg

$\Rightarrow \dfrac{|BD|}{|DP|} = \dfrac{|AN|}{|AP|}$

we also have

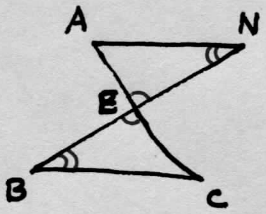


By same arg $\dfrac{|CD|}{|DP|} = \dfrac{|AM|}{|AP|}$

divide these

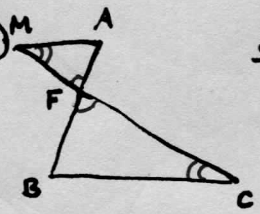
$$\frac{\frac{BD}{\cancel{DP}}}{\frac{CD}{\cancel{DP}}} = \frac{\frac{AN}{\cancel{AP}}}{\frac{AM}{\cancel{AP}}} \Rightarrow \frac{BD}{CD} = \frac{AN}{AM}$$

③



By same arg $\dfrac{|CE|}{|EA|} = \dfrac{|BC|}{|AN|}$

④



same arg $\dfrac{|AF|}{|FB|} = \dfrac{|AM|}{|BC|}$

⑤ Then form the product $\dfrac{|BD|}{|DC|}\,\dfrac{|CE|}{|EA|}\,\dfrac{|AF|}{|FB|} = \dfrac{\cancel{AN}}{\cancel{AM}}\,\dfrac{\cancel{BC}}{\cancel{AN}}\,\dfrac{\cancel{AM}}{\cancel{BC}} = 1$

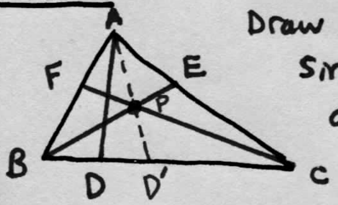
⑥ Now make the magnitudes signed: Since in each case D, E, F are between the side endpts going, for example B to D and D to C is going only 1 direction down the side, so we call this pos

$$\Rightarrow \frac{\overrightarrow{BD}}{\overrightarrow{DC}}\,\frac{\overrightarrow{CE}}{\overrightarrow{EA}}\,\frac{\overrightarrow{AF}}{\overrightarrow{FB}} = +1 \qquad \square$$

($\Leftarrow$) Pts D, E, F are given

Lines $\overline{CF}$ and $\overline{BE}$ meet at some pt inside Δ. Call it P.

We assume $\overline{AD}$ does NOT go thru P, but still have $\dfrac{BD}{DC}\,\dfrac{CE}{EA}\,\dfrac{AF}{FB} = 1$



Draw a line $\overline{AP}$ and extend it to the base at D′.
Since these 3 line segs do meet at P, first half of thm applies, and $\dfrac{BD'}{D'C}\,\dfrac{CE}{EA}\,\dfrac{AF}{FB} = 1$

cont'd →

Ceva's Pt Thm cont'd

Thus we have $\frac{BD'}{D'C}\frac{\cancel{CE}}{\cancel{EA}}\frac{\cancel{AF}}{\cancel{FB}} = 1 = \frac{BD}{DC}\frac{\cancel{CE}}{\cancel{EA}}\frac{\cancel{AF}}{\cancel{FB}} \Rightarrow \frac{BD'}{D'C} = \frac{BD}{DC}$

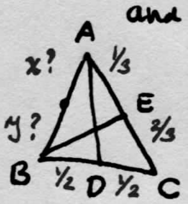
$$\Rightarrow \frac{BD'}{D'C} + 1 = \frac{BD}{DC} + 1 \Rightarrow \frac{BD'}{D'C} + \frac{D'C}{D'C} = \frac{BD}{DC} + \frac{DC}{DC}$$

$$\frac{\overbrace{BD' + D'C}^{BC}}{D'C} = \frac{\overbrace{BD + DC}^{BC}}{DC} \Rightarrow D'C = DC$$

D & D' are on the same side of C and same dist from C

$\Rightarrow D' = D$ $\boxed{\text{QED}}$

Remark In Ceva's pt thm, we could freely chose D, E and then F would be determined:



$\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} \overset{!}{=} 1 \Rightarrow \frac{\cancel{1/2}}{\cancel{1/2}}\frac{2/3}{1/3}\frac{x?}{y} = 1$

$\Rightarrow 2\frac{x}{y} = 1$ $\frac{x}{y} = \frac{1}{2}$ $x = 1/3$, $y = 2/3$ since $x + y = 1$ to account for whole length of side AB.

It is said Ceva and Menelaus are equivalent, and one can be proved from the other.

ch 2.4 Applications of Ceva & Menelaus Thms

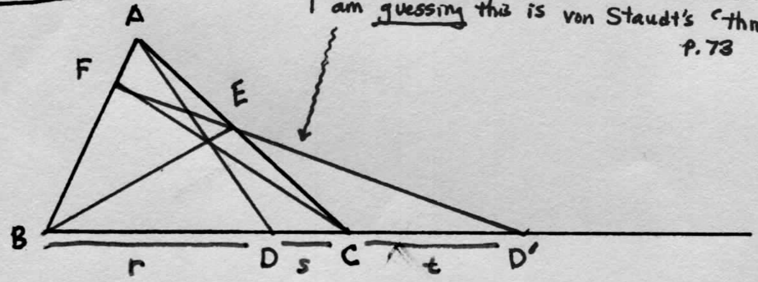
Thm (Harmonic Conj thm)
Consider ${}_B\overset{A}{\Delta}{}_C$ with side pts D, E, F making a Cevian pt configuration.
Extend segment $\overline{FE}$ so it meets (*) the extended base $\overline{BC}$ at some pt $D' =: h_{BC}(D)$

Menelaus line

$\Rightarrow$ D and D' divide $\overline{BC}$ in same ratio: $\cancel{\frac{DC}{BD}} = \cancel{\frac{CD'}{BD'}}$ $\boxed{\frac{BD}{DC} = \frac{-BD'}{D'C}}$ This is the form to use

(Later (p.82) we will know D' is harmonic conj of D wrt $\overline{BC}$)

I am guessing this is von Staudt's 'throw'? p.73



(*) In Proj Geo, we can consider FE $\parallel$ BC and meeting at ∞

By Ceva's Pt Thm $\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = +1$
(signed segment lengths but I'm not denoting it)

This is the thing, see p.56 def of Eves

and since we created $\overline{FED'}$ to be a Menelaus line $\frac{BD'}{D'C}\frac{CE}{EA}\frac{AF}{FB} = -1$

Combining these: $\frac{BD}{DC}\frac{\cancel{CE}}{\cancel{EA}}\frac{\cancel{AF}}{\cancel{FB}} = -\frac{BD'}{D'C}\frac{\cancel{CE}}{\cancel{EA}}\frac{\cancel{AF}}{\cancel{FB}} \Rightarrow \boxed{\frac{BD}{DC} = \frac{-BD'}{D'C}}$

In terms of Cross Ratio (coming up p.73)

$$\frac{BD}{DC}\frac{D'C}{BD'} = -1$$

reverse engineer CR

$$(BC, DD') = -1$$

This is called harmonic p.82

D' is harmonic conj of D wrt $\overline{BC}$ $\square$

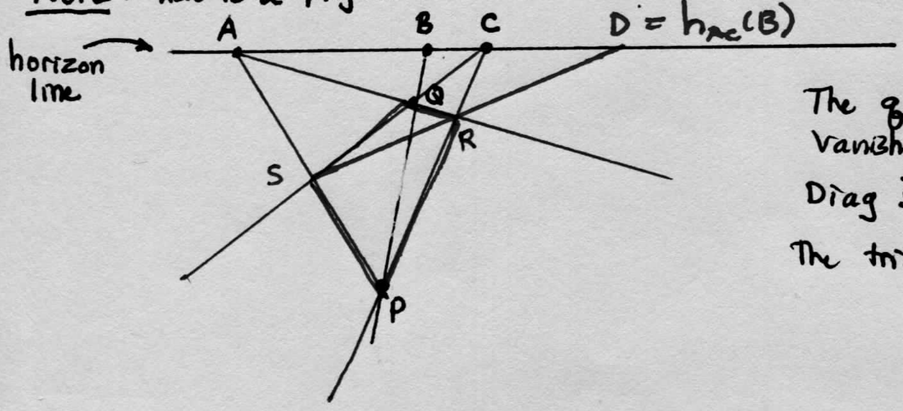
reciprocating, and using $-D'C = CD'$ $\Rightarrow \frac{DC}{BD} = \frac{CD'}{BD'}$ or $\frac{s}{r+s} = \frac{t}{r+s+t}$

This is something, but unrelated to how Eves defines things. $\square$

Eves ASOG
P. 68

1/26/2025
6

Vijay
NOTE: Here is a Projective Geo version from youtube 'Geo of Vision' Ravikumar § 7D



The quad PRQS has 2 sides going to Vanishing pt A, and 2 to vanishing pt C
Diag $\overline{SR}$ meets horiz at D
The triang ${}_{A}\overset{P}{\Delta}{}_{C}$ corresponds to prev triang ${}_{B}\overset{A}{\Delta}{}_{C}$

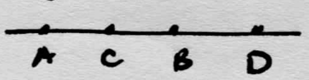
Skipping for Now
- Desargues Thm
- Pascal's Mystic Hexagram Thm

2/1/2025

Eves p. 82-83

A, B, C, D pts on a line, in some order

$(AB, CD) = -1$ so we know $\overline{AB}$ and $\overline{CD}$ are the segments and from neg value, we know they are interlaced

e.g. 

They say this means the words:

"C, D divide $\overline{AB}$ one internally, one ext, in the same numerical ratio"

"harmonic conjugates"

"C & D divide $\overline{AB}$ harmonically"

Trying to make sense of these words is a fool's errand.

$$\boxed{\begin{aligned} h_{AB}(C) &= \frac{AC}{CB} \\ h_{AB}(D) &= \frac{AD}{DB} \end{aligned}}$$

$$(AB, CD) = \frac{\frac{AC}{CB}}{\frac{AD}{DB}} = \frac{AC}{CB}\frac{DB}{AD} = -1 \quad \text{or} \quad \frac{AC}{CB} = -\frac{AD}{DB}$$

and $\frac{AD}{DB}$ is inherently neg since D external

$$h_{AB}(C) = -h_{AB}(D)$$

Thm 2.8.2 "C & D divide $\overline{AB}$ harmonically" $\Leftrightarrow$ "A & B divide $\overline{CD}$ harmonically"

i.e. $(AB, CD) = -1$ $\qquad (CD, AB) = -1$

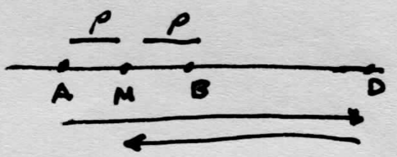
pf

This follows from interval position sym of CR: $(AB, CD) = (CD, AB)$ $\square$

Thm 2.8.3 Let M be midpt of $\overline{AB}$ $\Rightarrow$ Harmonic conj wrt AB of M is ∞

$(AB, M\infty) = -1$

pf. Let D be unknown conj pt of M



$$(AB, MD) = -1 \Rightarrow \frac{AM}{MB} \cdot \frac{DM}{AD} = -1$$

$$\Rightarrow \frac{\overrightarrow{AM}}{\overrightarrow{MB}} = -\frac{\overrightarrow{AD}}{\overrightarrow{DM}}$$

$x := \text{dist}(D, M)$

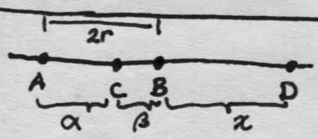
$$h_{AB}(M) = \frac{AM}{MB} = \frac{m-a}{b-m} = \frac{\rho}{\rho} = {}^{+}1 \quad \text{Thus} \quad h_{AM}(D) \overset{!}{=} -1$$

$$\frac{AD}{DM} = \frac{d-a}{m-d} = \frac{x+\rho}{-x} \longrightarrow -1 \text{ as } x \to \infty$$

$\Rightarrow D =$ pt at ∞ $\square$

My Thm Given fixed interval $\overline{AB}$, pt C between A, B $\Big\}$ $\Rightarrow$ Always $\exists$ pt D beyond B $\ni (AB, CD) = -1$

pf.



$$\frac{AC}{CB}\frac{DB}{AD} = -1$$

$$\frac{\alpha}{\beta}\frac{(-x)}{2r+x} = -1$$

$$\alpha x = \beta(2r + x) = 2r\beta + \beta x$$

$$(\alpha - \beta)x = 2r\beta$$

$$x = \frac{2r\beta}{(\alpha-\beta)} = \frac{2r\beta}{2r-\beta-\beta} = \frac{r\beta}{r-\beta}$$

$\alpha + \beta = 2r$ (related to diam of circle of inversion in next chapter)

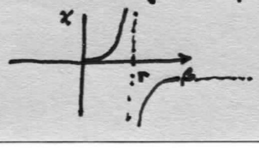
$\alpha = 2r - \beta$

Here I am specifically considering $0 < \beta < 2r$ but we could expand to $\beta \in \mathbb{R}$

This always has a soln

$\beta = 0$ ($C = B$) $\Rightarrow$ $x = 0$ ($D = B$)

$\beta = r$ (C midpt) $\Rightarrow$ take $x = \infty$



$\beta = 2r \Rightarrow x = -2r$

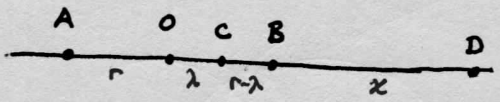
$C = A \Rightarrow D = A$

$\square$

2/9/2025

Eves p. 82–83 addendum

Let's continue the same discussion with fixed interval AB and explore further the relation between $(AB, CD) = -1$ and circle inversion (which Eves hasn't introduced yet) but Thm 2.8.5 is relevant. p. 83–84



$$\frac{AC}{CB}\,\frac{DB}{AD} \overset{!}{=} -1$$

$$\frac{(r+\lambda)}{(r-\lambda)}\,\frac{(-x)}{(2r+x)} = -1 \quad \Rightarrow \quad -(r+\lambda)x = -(r-\lambda)(2r+x)$$

$$(r+\lambda)x = 2r(r-\lambda) + x(r-\lambda)$$

$$(r+\lambda)x - (r-\lambda)x = 2r(r-\lambda)$$

$$2\lambda x =$$

$$x = \frac{r(r-\lambda)}{\lambda} \qquad \lambda = 0 \quad x = \infty \quad (C = O)$$

$$\lambda = r \quad x = 0 \quad (C = B)$$

The dist of C from O is λ

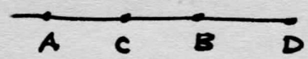
Dist of D from O is $r + x =: \mu$

$$\mu = r + \frac{r(r-\lambda)}{\lambda} = \frac{\lambda r + r(r-\lambda r)}{\lambda} = \frac{r^2}{\lambda}$$

Thus $OC \cdot OD = \lambda\mu = \lambda\left(\frac{r^2}{\lambda}\right) = r^2$ and this is precisely the inversion condition. Eves introduces it later.

Thm 2.8.4 $(AB, CD) = -1 \iff 2 \cdot \frac{1}{\overline{AB}} = \frac{1}{\overline{AC}} + \frac{1}{\overline{AD}}$

This is the perpetual abuse of notation confusing the directed segment w/ the length (signed) of the segments.



Pf. $\frac{AC}{CB}\frac{DB}{AD} = -1 \Rightarrow \frac{AC}{CB} = \frac{-AD}{DB} \xrightarrow{\text{invert}} \frac{CB}{AC} = \frac{\overset{+BD}{-DB}}{AD}$

mult both sides by $\frac{1}{AB}$: $\frac{CB}{AB \cdot AC} = \frac{BD}{AB \cdot AD}$

By Thm 2.1.3 Cyclic segment sum $CB = AB - AC$, $BD = AD - AB$

because (A → B → C) $AB + BC + CA = 0$
$BC = -AB - CA$
$-CB = -AB + AC$
$CB = AB - AC$

likewise (A → B → D)

$$\Rightarrow \frac{AB - AC}{AB \cdot AC} = \frac{AD - AB}{AB \cdot AD}$$

$$\frac{\cancel{AB}}{\cancel{AB} \cdot AC} - \frac{\cancel{AC}}{AB \cdot \cancel{AC}} = \frac{\cancel{AD}}{AB \cdot \cancel{AD}} - \frac{\cancel{AB}}{\cancel{AB} \cdot AD} \Rightarrow \frac{1}{AC} - \frac{1}{AB} = \frac{1}{AB} - \frac{1}{AD}$$

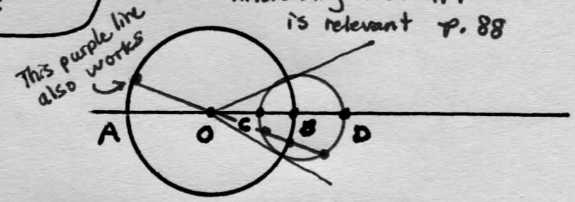
$$\Rightarrow \frac{1}{AC} + \frac{1}{AD} = \frac{2}{AB} \qquad \square$$

Thm 2.8.5 Let O be midpt of $\overline{AB}$

$(AB, CD) = -1 \iff OB^2 = OC \cdot OD$

circle inversion $r^2 = OC \cdot OC'$

This picture of 2 circles intersecting O.G. is relevant p. 88



($\Rightarrow$) From prev work above $\frac{AC}{CB} = \frac{-AD}{DB}$

Also above $AC = OC - OA$ $AD = OD - OA$
$CB = OB - OC$ $DB = OB - OD$

$OB = -OA$ since O midpt

$$\Rightarrow \frac{OC - OA}{OB - OC} = \frac{-(OD - OA)}{OB - OD} \Rightarrow \frac{OC + OB}{OB - OC} = \frac{OD + OB}{OD - OB}$$

$$\Rightarrow (OC + OB)(OD - OB) = (OD + OB)(OB - OC)$$

$$OC \cdot OD - \cancel{OC \cdot OB} + \cancel{OB \cdot OD} - OB^2 = \cancel{OD \cdot OB} - OD \cdot OC + OB^2 - \cancel{OB \cdot OC}$$

$$2\, OC \cdot OD = 2\, OB^2$$

$$OC \cdot OD = OB^2$$

($\Leftarrow$) reverse these steps

$\square$

Def a seq (b_n) is an arithmetic progression if $b_{n+1} = b_n + r$ for some fixed r

Seq can be finite here

i.e. $b_{n+1} = b_0 + (n)r = b_0 + nr$

$\{1, 2, 3, 4, \dots\}$ has $b_0 = 1$ and $r = 1$

$\{5, 7, 9, 11, \dots\}$ has $b_0 = 5$ and $r = 2$

Def a seq (a_n) is a harmonic progression if $a_n = \frac{1}{b_n}$ where (b_n) is arithmetic prog.

Thm 2.8.7 seq $\{a_1, a_2, a_3\}$ is harmonic $\Leftrightarrow$ $\frac{1}{a_1} + \frac{1}{a_3} = \frac{2}{a_2}$

Pf. $(\Rightarrow)$ $a_1 = \frac{1}{b_0+r}$ $\quad a_2 = \frac{1}{b_0+2r}$ $\quad a_3 = \frac{1}{b_0+3r}$

$b_1 + b_3 = 2b_2 \qquad b_2 = \frac{b_1+b_3}{2}$

$b_p + b_{p+2}$

$$\frac{1}{a_1} + \frac{1}{a_3} \stackrel{?}{=} \frac{2}{a_2}$$

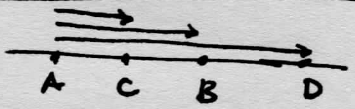
$$b_0+r + b_0+3r \stackrel{?}{=} 2(b_0+2r)$$

$$4r = 4r \checkmark$$

$(\Leftarrow)$ same idea in reverse

$\boxed{\text{Thm 2.8.8} \quad (AB, CD) = -1 \Rightarrow \{\underset{a_1}{AC}, \underset{a_2}{AB}, \underset{a_3}{AD}\} \text{ is harmonic prog}}$

pf. CR being neg means intervals are interlaced



The claim is $\frac{1}{AC}, \frac{1}{AB}, \frac{1}{AD}$ is b_1, b_1+r, b_1+2r

$$\frac{AC}{CB}\frac{DB}{AD} = -1 \overset{\text{Thm 2.8.4}}{\Longleftrightarrow} \underset{a_2}{\frac{2}{AB}} = \underset{a_1}{\frac{1}{AC}} + \underset{a_3}{\frac{1}{AD}}$$

$$\frac{2}{a_2} = \frac{1}{a_1} + \frac{1}{a_3} \Rightarrow \text{Thm 2.8.7} \ \{\overset{\frac{1}{2}}{a_1}, \overset{\frac{1}{3}}{a_2}, \overset{\frac{1}{4}}{a_3}\} \text{ is harm}$$

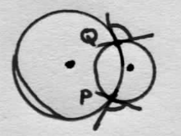
$$\Rightarrow \{\underset{\frac{1}{b_0+r}}{AC}, \underset{\frac{1}{b_0+2r}}{AB}, \underset{\frac{1}{b_0+3r}}{AD}\} \text{ is harmonic}$$

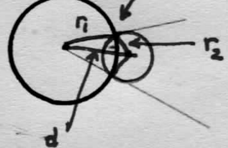
So no info on specific values for

Eves p. 88 2/2/2025

Thm 2.9.2 (1) circles C_1 and C_2 intersect at pts P, Q $\Rightarrow$ The angles of intersection are the same at P & Q.

(2) C_1 and C_2 intersect O.G. $\quad C_1 \pitchfork C_2 \quad \Leftrightarrow$ The radial segment of C_1 is tangent to C_2 (and vice versa) at pts of intersection





(3) $C_1 \pitchfork C_2 \Leftrightarrow d^2 = r_1^2 + r_2^2$
(just Pythag Thm)

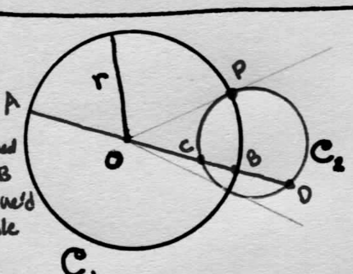
(4) $C_1 \pitchfork C_2 \Rightarrow$ The center of either lies outside the other
$d > r_1$
$d > r_2$ (hypot greater than either side)

Pf. Only (1) requires some comments since Hartshorne p. makes it seem hard.
We know a circle is reflection sym wrt any diam.
The line of centers extends to encompass diam lines for both C_1 and C_2
Therefore the assembly $C_1 \cup C_2$ is sym wrt this line and P is reflection of Q
so the angles at P and Q are the same □

Coxeter ITG 2nd p.88

Thm 2.9.3 $C_1 \pitchfork C_2 \Leftrightarrow$ any diam line of C_1 that is cut by C_2 is cut in a harmonic division i.e. $(AB, CD) = -1$

Pf.



If we turned the line AB about O, we'd get a whole bunch of harmonic divisions!

($\Rightarrow$) Let P be a pt where C_1 and C_2 meet. Thus $OP = r$
By above Thm 2.9.2 (2) $\overline{OP}$ is tangent to C_2
By Power of pt O wrt C_2 $\quad OP^2 = OC \cdot OD$ — Eves doesn't introduce this until p. 90
OB is also a radial seg of C_1 so
$OB^2 = OC \cdot OD$
O is midpt of $\overline{AOB}$ so Thm 2.8.5 gives us $(AB, CD) = -1$

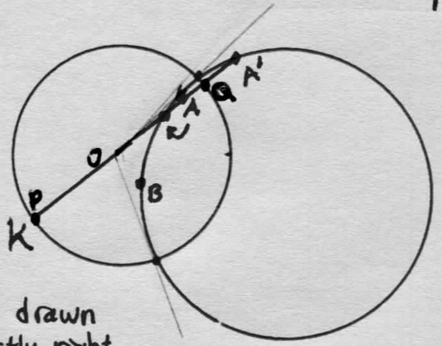
($\Leftarrow$) C_1 has diam line $\overline{AOB}$ cut harmonically by C_2, that is $(AB, CD) = -1$
This time let $P \in C_2$ be such that $\overline{OP}$ is tangent to C_2
By Power of pt O wrt C_2 $\quad OP^2 = OC \cdot OD$
Again O is midpt of AOB so Thm 2.8.5 gives $OB^2 = OC \cdot OD = OP^2$
$(AB, CD) = -1$
Thus $OP = r$ since $OB = r$ by construction of B
$\Rightarrow P \in C_1$ also
The pt P where C_2 meets C_1 is also a pt where radial seg $\overline{OP}$ of C_1 is tangent to C_2
$\Rightarrow C_1 \pitchfork C_2$

□

Thm 2.9.5 Given circle K, pts A, B in disc interior $\Big\} \Rightarrow$ $\exists$ exactly one circle/line $C \ni$
(1) $C \pitchfork K$
(2) C passes thru A & B

pf. (1) If O, A, B lie on a line, then this line is C
this line is uniq.

(2) O, A, B not on a line
Extend $\overline{OA}$ to $\overline{PQ}$ a diameter and then still further to some pt A' so that CR $(PQ, AA') = -1$
How do we know this can be done? That A' exists?



Not drawn exactly right

From my discussion on p.82-83 sheets, for a fixed interval $\overline{PQ}$ and a pt A between them, there always exists a pt A' beyond $Q \ni (PQ, AA') = -1$. A' is uniq.

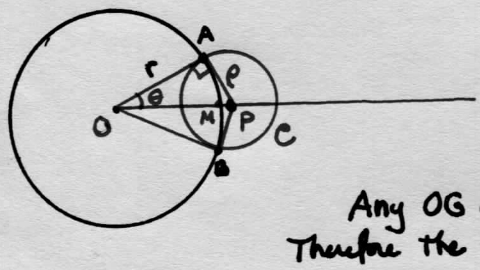
A & A' are 'harmonic conjugates'

(3) Let C be the circle defined by the 3 pts A, A', B
We know C is OG to K by prev 2.9.3 Thm

(4) Show uniqueness
we already know uniqueness when C is a line
For C_2 ~~from step (2)~~ being a circle, $C_2 \pitchfork K$
C_2 passes thru A, B
$\overline{POAQ}$ is a diam line, and cut by C_2 at A
Thm 2.9.3 Says this diam line (when extended) is cut harmonically $(PQ, AZ) = -1$ by C_2 for some Z beyond Q on the line. But since we already know $(PQ, AA') = -1$ we must have $Z = A'$. And since a priori $Z \in C_2$, $A' \in C_2$
These are the same 3 pts as before and $C_2 = C$ $\square$

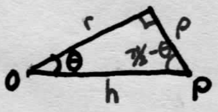
My Thm: Given circle K, pts A, B on K $\Rightarrow$ $\exists!$ a circle/line C such that
(1) $C \pitchfork K$
(2) C meets K at A & B

pf. (a) If $\overline{AB}$ is a diam of K, then the line $\overleftrightarrow{AB}$ is C



(b) In the case AB is not a diam, let M be the midpt on the smaller arc from A to B.
Consider the ray $\overrightarrow{OM}$. The circle K and pts A, B are reflection sym about this axis.
Let $\theta := \angle_{O\,M}^{\;A}$

Any OG circle C must meet K at A & B and be tangent to $\overrightarrow{OA}$ at A, $\overrightarrow{OB}$ at B
Therefore the center P and radius ρ are determined from trig.



$\cos\theta = \frac{r}{h} \Rightarrow h = \frac{r}{\cos\theta}$ location of center P

Radius $\rho = r \tan\theta$

Thus C is determined, and uniq by center and radius $\square$

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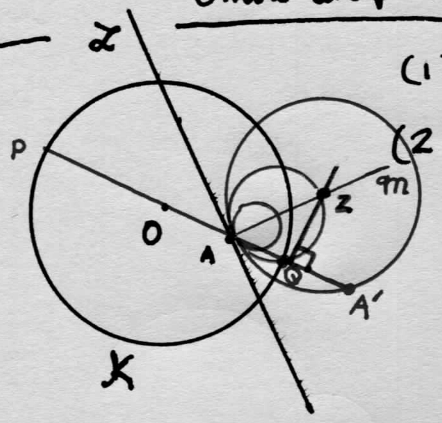
Thm 2.9.6 Given circle K and line $\mathcal{L}$. Choose arb pt $A \in \mathcal{L}$, $A \notin K$ $\Rightarrow$ $\exists$ a uniq circle/line C that is
(1) $C \perp K$
(2) C is tangent to $\mathcal{L}$ at A



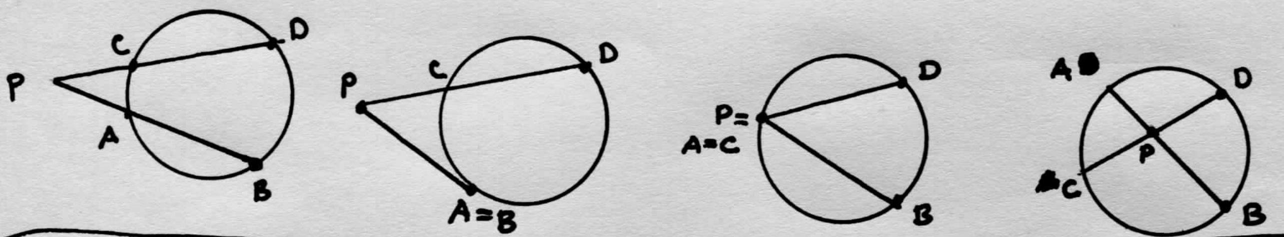
(1) If $O \in \mathcal{L}$ then $C = \mathcal{L}$ since it is a diameter line

(2) $O \notin \mathcal{L}$

There is a line $\mathcal{M}$ that meets $\mathcal{L}$ at A and is $\perp$
Any circle tanget to $\mathcal{L}$ at A must have center on $\mathcal{M}$
(In the configuration shown, C lies on opposite side of $\mathcal{L}$ than O.
For any candidate circle tangent at A, extending $\overline{OA}$ to $\overrightarrow{POAQ}$ this ray must cut the candidate circle twice.
Choose as C the one where $CR(PQ, AA') = -1$
AA' is a chord of C. We can draw its $\perp$ bisector and we know that this must pass thru the center of C (cf Sheet 31 of my 'Euclidean Geo' - writeup of Jacobs)
where this line intersects $\mathcal{M}$ (pt Z) is the center of C and radius is $|AZ|$. □

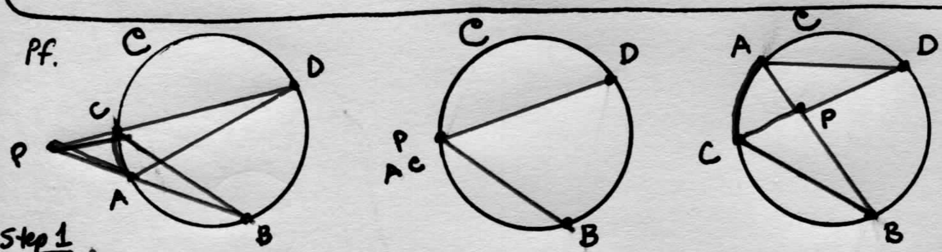
▷ Consider a fixed circle C and a pt P that is in various relative positions.
There is an invariant quantity $J_C(P)$ (Steiner's Power of pt P wrt circle C)
Given P, it has the same value on all lines thru P that cut C

Following Jacobs Geo Sheet 36 Secant lines



Thm Given C and P. Given 2 arb lines from P cutting C at A,B and C,D resp $\Rightarrow$ $\overrightarrow{PA}\cdot\overrightarrow{PB} = \overrightarrow{PC}\cdot\overrightarrow{PD}$ (signed distances)
Thus define $J_C(P) := \overrightarrow{PA}\cdot\overrightarrow{PB}$

Pf.



w.l.o.g. I am keeping D, B fixed and letting pt P move inside.

Step 1

$\Delta_{PAD} \sim \Delta_{PCB}$ always, whether P is inside or out (or on C in a collapsed state)
$\measuredangle_P$ is always the same: P outside, both Δs share same physical angle
P inside, both have a vertex-ical angle at P, equal values
$\measuredangle_D$ is always inscribed angle $\frac{1}{2}$ m $\widehat{AC}$, even when A and C switch places. The same is true for $\measuredangle_B$
$\Rightarrow \measuredangle_D = \measuredangle_B$ always. The result follows by (A.A. Similarity)

Cont'd →

Eves p.89 cont'd

2/15/2025

Step 2 The opp side of $\measuredangle_P$ is $\overline{AD}$ and $\overline{CB}$, resp. $\Rightarrow$ $|AD| = \lambda |CB|$

opp side of $\measuredangle_D$ is PA
opp side of $\measuredangle_B$ is PC $\Rightarrow$ $|PA| = \lambda |PC|$

And for the only side left $|PD| = \lambda |PB|$

Step 3 Form the quantity $|PA|\,|PB| = \cancel{\lambda} |PC| \cdot \frac{1}{\cancel{\lambda}} |PD|$

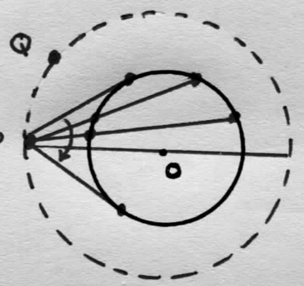
Step 4 To go to "signed lengths", when P is outside $\overrightarrow{PA}$ and $\overrightarrow{PB}$ are in the same direction, and so are $\overrightarrow{PC}$ and $\overrightarrow{PD}$ so both either pos or neg together, depending on $\mathcal{O}^R$ of lines

$PA \cdot PB = PC \cdot PD$ $J_C(P)$ is pos

For P on C itself $PA = 0 = PC$ so again $J_C(P) = 0$

For P inside C, $\overrightarrow{PA}$ and $\overrightarrow{PB}$ always point opposite dirs, so $J_C(P)$ is neg $\square$

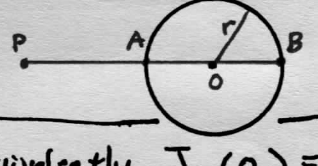
Thus $J_C(P)$ is const (has the same value for line segment products) in all these red lines:



In fact J_C has the same value for all pts for Q on circle with radius $|OP|$ ⊛

Thm 2.9.9 $J_C(P) = |OP|^2 - r^2$

pf. choose ray $\overrightarrow{PO}$. Then $J_C(P) = PA \cdot PB = (PO - r)(PO + r) = PO^2 - r^2$ $\square$

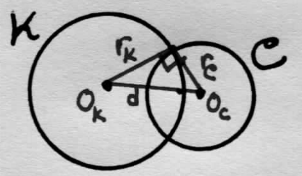


Thm 2.9.10 Circles $K \pitchfork C \Leftrightarrow J_C(O_K) = r_K^2$ (equivalently $J_K(O_C) = r_C^2$)

Pf. ($\Rightarrow$) Since circles OG, by Thm 2.9.2 (3) p.88

$d^2 = r_K^2 + r_C^2$ and $d = |O_K O_C|$

and $J_C(O_K) = |O_K O_C|^2 - r_C^2$
$= d^2 - r_C^2$
$= r_K^2 + r_C^2 - r_C^2 = r_K^2$ ✓



($\Leftarrow$) $J_C(O_K) = r_K^2$
$|O_K O_C|^2 - r_C^2 = r_K^2$
$d^2 = r_K^2 + r_C^2 \Rightarrow K \pitchfork C$ by Thm 2.9.2(3)

$\square$

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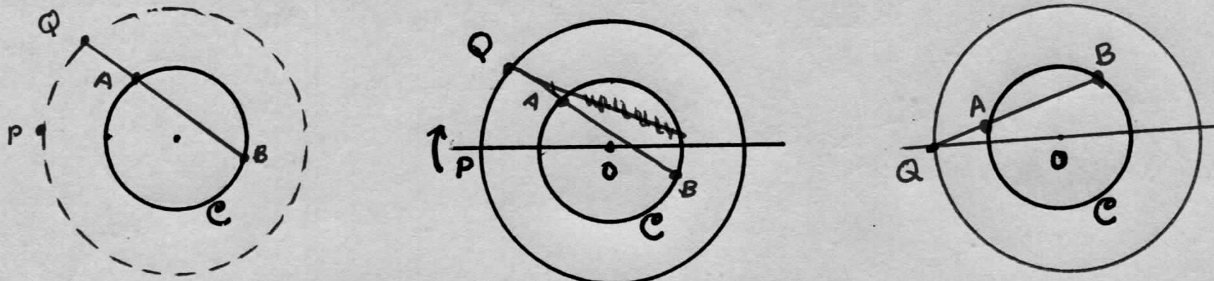
▷ Before moving on, let's attempt a <u>Sym arg</u> for (*) on prev sheet
J_C has the same value $\forall\, Q \in$ (Circle with radius $|OP|$)



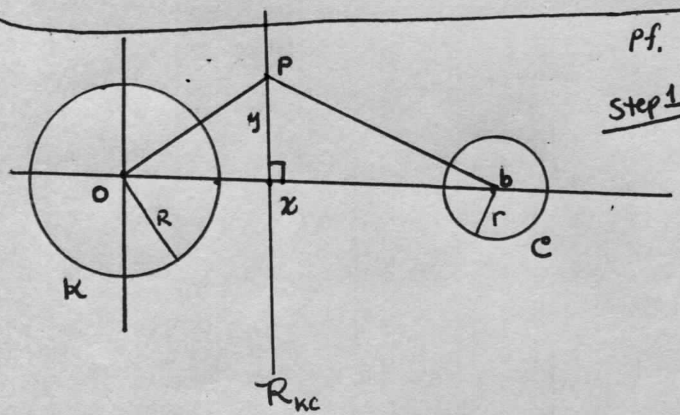
We know the result is true for any line cutting C from pt P.
But what if we chose pt Q and draw a line cutting C at A and B?
Define a reference line $\overline{PO}$ (i.e. a co-ord sys) this is the x-axis.
Rotate the <u>co-ord sys</u> so the x axis overlays $\overline{QO}$. Then the line $\overline{QAB}$ looks just like the prev case of a line from P. C looks identical; by axiom, all distances measured in the co-ord sys are unchanged by rotation. Thus $J_C(P) = J_C(Q)$.
Do we need a co-ord system? without one, rotation wrt what?

[We could just say $J_C(Q) = J_C(P)$ because P was arb, and line segs from Q have same distances]

▷ <u>Ch 2.10</u> Radical Axis of 2 Circles

p.92 Given 2 circles K and C, define Radical axis $R_{KC} := \left\{ \underset{x}{\text{all pts}} \;\middle|\; J_K(x) = J_C(x) \right\}$

<u>Thm 2.10.2</u> Circles K and C not concentric $\Rightarrow$ R_{KC} is a straight line $\perp$ to the line of centers between K and C



Pf. I am introducing a co-ord sys with origin at center of K. Eve's pf is complex and I'm not sure it does it.
<u>Step 1</u> Is there a pt on the x-axis where $J_K(x) = J_C(x)$?

$$x^2 - R^2 \overset{!}{=} (b-x)^2 - r^2 \quad (*)$$
$$\cancel{x^2} - R^2 = b^2 - 2bx + \cancel{x^2} - r^2$$
$$2bx = R^2 - r^2 + b^2$$
$$x = \frac{b^2}{2b} + \frac{R^2 - r^2}{2b} = \frac{1}{2}b + \overbrace{\frac{R^2 - r^2}{2b}}^{\text{shift of midpt}}$$

If sym circles ($R = r$) the $x = \frac{1}{2}b$

<u>Step 2</u> Now let's go up from x a dist y. Does this pt P satisfy $J_K(P) \overset{?}{=} J_C(P)$

$$|OP|^2 - R^2 \overset{?}{=} |Pb|^2 - r^2$$
$$x^2 + \cancel{y^2} - R^2 = (b-x)^2 + \cancel{y^2} - r^2$$ this is just (*) again

but $s \neq x$ from step 1?

<u>Step 3</u> Can there be any pt not on this line? that is $\langle s,t \rangle \in R_{KC}$

$$(s^2 + \cancel{t^2}) - R^2 \overset{?}{=} (b-s)^2 + \cancel{t^2} - r^2$$

reduces just like step 1 to $s = \frac{b}{2} + \frac{R^2 - r^2}{2b} = x$

□