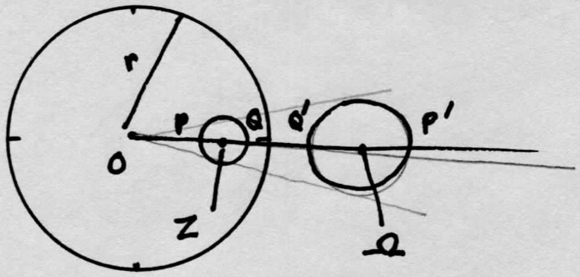
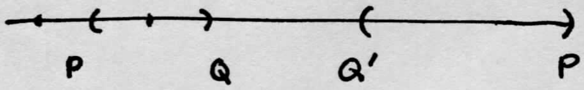
If a circle $\mathcal{C}$ is contained in K, show that the center of $\mathcal{C}$ is NOT mapped to the center of $\mathcal{C}'$.



The center of $\mathcal{C}$ (call it Z) is mapped along the radial line $\overrightarrow{OZ}$ $\ni$ $OZ \cdot OZ' = r^2$ but the center of $\mathcal{C}'$ (call it Ω), while on line $\overrightarrow{OZ}$, is not Z'.

One dim
Here P, Q also denote distances on Real line



mid pt $Z = \frac{P+Q}{2}$ midpt $\Omega = \frac{Q'+P'}{2} = \frac{\frac{r^2}{Q} + \frac{r^2}{P}}{2} = \frac{r^2}{2}\left(\frac{1}{\cancel{Q}} + \frac{1}{P}\right)$

But $Z' = f(Z) = \frac{r^2}{\frac{P+Q}{2}} = 2r^2\left(\frac{1}{P+Q}\right)$

So we'd have to have $2\cancel{r^2}\left(\frac{1}{P+Q}\right) = \frac{\cancel{r^2}}{2}\left(\frac{1}{Q} + \frac{1}{P}\right) \Rightarrow 4\left(\frac{1}{P+Q}\right) = \frac{1}{Q} + \frac{1}{P}$

$\frac{4}{2P} \overset{?}{=} \frac{1}{P} + \frac{1}{P}$

$2\left(\frac{1}{P}\right) = \frac{2}{P}$

We know the center IS preserved when $\mathcal{C}$ meets K OG
or is it?

$\frac{4}{p+q} = \frac{1}{q} + \frac{1}{p} = \frac{p+q}{pq}$

$4pq = (p+q)^2 = p^2 + 2pq + q^2$

$0 = p^2 - 2pq + q^2$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$p = \frac{+2q \pm \sqrt{4q^2 - 4 \cdot 1 q^2}}{2}$

$= \frac{2q}{2} = q \Rightarrow p = q$

So $Z' = \Omega \Leftrightarrow$ circle is a pt, radius $= 0$
which means it never happens.

Ch 25 Consistency of HB Postulates 12/23/2017

Inversion

$f: \mathbb{E} - \{A\} \longrightarrow \mathbb{E} - \{A\}$

$P \longmapsto P'$ such that $|AP'| = \frac{a^2}{|AP|}$

In polar co-ords

$f: \mathbb{E} - \{0\} \longrightarrow \mathbb{E} - \{0\}$

$(r, \theta) \longmapsto \left(\frac{a^2}{r}, \theta\right)$

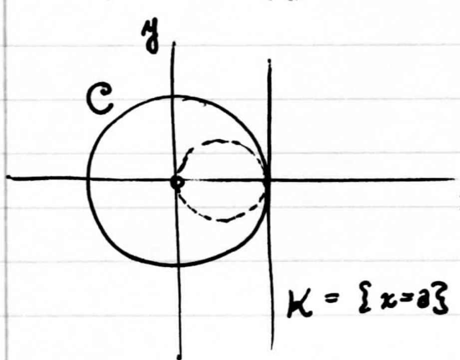
Good!

Rectangular

$(x, y) \longmapsto$

$x = r\cos\theta$
$y = r\sin\theta$
$r = \sqrt{x^2+y^2}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

p. 420



$x = a$

$r\cos\theta = a$ $\qquad r = \frac{a^2}{s}$

$\frac{a^2}{s}\cos\theta = a \qquad s \neq 0$

$a\cos\theta = s \qquad s \neq 0$

$s^2 = x^2 + y^2$

$x = s\cos\theta$
$y = s\sin\theta$

θ

same idea
more generalized

$s^2 = as\cos\theta$

$x^2 + y^2 = ax$

$x^2 - ax + y^2 = 0$ complete square

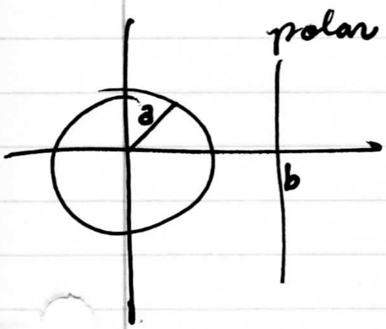
$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$ circle centered at $a/2$, radius $a/2$

$x^2 + -2x\frac{a}{2} + \frac{a}{2}$

Thm 5 K is a line in $\mathbb{E} - \{0\} \Longrightarrow f(K)$ is a punctured circle.

pf. Lets say wlog $x = b$, $b > 0$

polar $r\cos\theta = b$



$\frac{a^2}{s}\cos\theta = b \quad s \neq 0$

$\frac{a^2}{b}\cos\theta = s$

$\frac{a^2}{b} s\cos\theta = s^2$

$\frac{a^2}{b} x = x^2 + y^2 \Rightarrow 0 = x^2 - \frac{a^2}{b}x + y^2$

$\left(\frac{a^2}{2b}\right)^2 = \left(x - \frac{a^2}{2b}\right)^2 + y^2$

center $= \frac{a^2}{2b}$ radius $\frac{a^2}{2b}$ $\square$

here he says $\mathcal{L}$ = circle
but $\mathcal{L}$ should mean Line
K means Circle

$f^2 = Id$

$f(f(P)) = P \Rightarrow f(\mathcal{L}) = f(f(K)) = K$

converse of Thales

Thm 7: M is a circle in $\mathcal{E} - \{A\} \Rightarrow f(M)$ is a circle in $\mathcal{E} - \{A\}$

M is the graph of $\underbrace{x^2 + y^2}_{} + Ax + By + C = 0 \quad C \neq 0$

polar form $r^2 + Ar\cos\theta + Br\sin\theta + C = 0$

$r = \frac{a^2}{s}$

f(M) is graph of $\frac{a^4}{s^2} + A\frac{a^2}{s}\cos\theta + B\frac{a^2}{s}\sin\theta + C = 0$

$$a^4 + Aa^2\underbrace{s\cos\theta}_{x} + Ba^2\underbrace{s\sin\theta}_{y} + Cs^2 = 0$$

$$a^4 + Aa^2x + Ba^2y + C(x^2+y^2) = 0$$

$$x^2 + y^2 + \frac{Aa^2}{C}x + \frac{Ba^2}{C}y + \frac{a^4}{C} = 0$$

This is the eq of a circle

we know this IS a circle

$$x^2 + \frac{Aa^2}{C}x + \left(\frac{Aa^2}{2C}\right)^2 + y^2 + \frac{Ba^2}{C}y + \left(\frac{Ba^2}{2C}\right)^2 = \left(\frac{Aa^2}{2C}\right)^2 + \left(\frac{Ba^2}{2C}\right)^2 - \frac{a^4}{C}$$

$$\Rightarrow \left(x + \frac{Aa^2}{2C}\right)^2 + \left(y + \frac{Ba^2}{2C}\right)^2 = \frac{(Aa^2)^2 + (Ba^2)^2}{4C^2} - \frac{a^4}{C}\frac{4C}{4C}$$

$$= \frac{A^2a^4 + B^2a^4 - 4Ca^4}{4C^2}$$

$$= \frac{(A^2+B^2-4C)a^4}{4C^2}$$

$x^2 + Ax + y^2 + By = -C \quad C \neq 0$

$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 = \frac{A^2+B^2}{4} - C$

$= \frac{A^2+B^2-4C}{4}$

so since we know we have a circle $A^2+B^2-4C > 0$

so this works

circle centered at $\left(\frac{A}{2}, \frac{B}{2}\right)$ rad $= \frac{A^2+B^2-4C}{4}$ $\longrightarrow$ circle centered $\left(\frac{A}{2}\frac{a^2}{C}, \frac{B}{2}\frac{a^2}{C}\right)$ rad $= \left(\frac{A^2+B^2-4C}{4}\right)\left(\frac{a^2}{C}\right)^2$

factor $\frac{a^2}{C}$

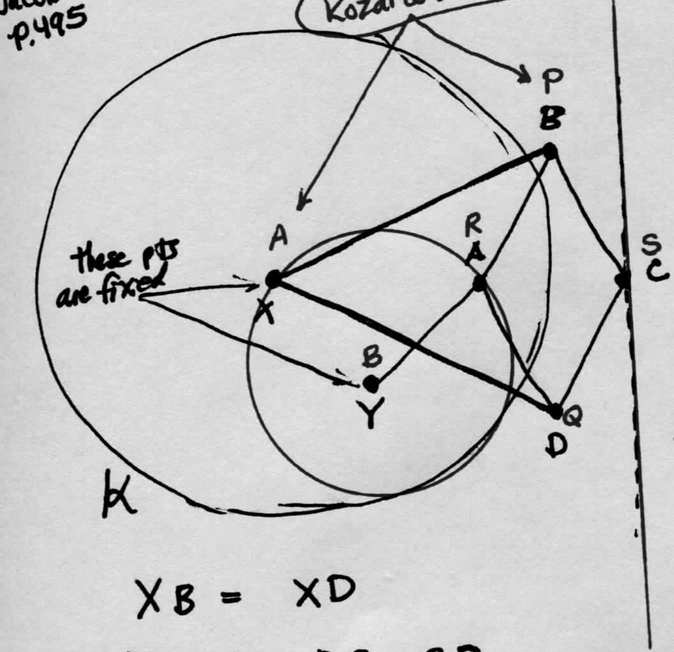
2/3/2018

Peaucellier Linkage

Can we build a mechanism such that one part moving in a circle causes another part to move in a straight line?

Names of pts in Kozai & Libeskind p.22

Jacobs p.495



$XB = XD$

$BA = AD = DC = CB$

(1.) why are pts X, A and C colinear?

X is always equi-dist from B and D

C " " " " " "

$\Rightarrow \overline{XC} \perp$ bisects $\overline{BD}$

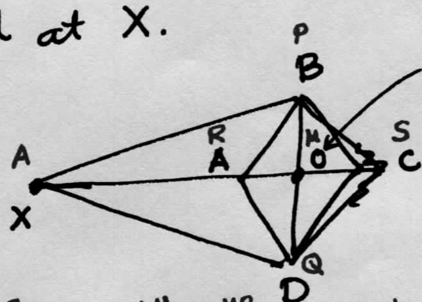
[Equidist pts make $\perp$ bisector thm Stur(8) Jacobs p.179]

likewise

$\overline{XA}$ and $\overline{AC}$ both $\perp$ bisect BD

The $\perp$ bisector line of $\overline{BD}$ is unique (for any given position of B, D) so X, A, C all colinear.

(2) show C is inverse of pt A wrt a circle K centered at X.



Let O be the unique bisect pt — i.e. midpt of $\overline{BD}$

$$\overset{AR\quad AS}{XA \cdot XC} = \overset{AM - MR}{[XO - OA]}\ \overset{AM + MS}{[XO + OC]}$$

since $OA = OC$

$$XA \cdot XC = [XO - OA][XO + OA]$$

$$= (XO)^2 - (OA)^2$$

Since we established $BD \perp AC$, we have right triangs and can apply pythag

$$= (XB)^2 - \cancel{(BO)^2} - [BA^2 - \cancel{BO^2}]$$

$$= (XB)^2 - (BA)^2$$

Both of these lengths are fixed indep of where the moveable pts are.

Thus the radius of the circle K is $\sqrt{(XB)^2 - (BA)^2}$

Why is this pos? Just take $|XB| > |BA|$

Now by affixing another bar $\overline{AY}$ to a fixed pt Y, A must move in a circle of radius $|AY|$ and thus since A and C are inverse pts C must move in a line.

$\boxed{\text{QED}}$

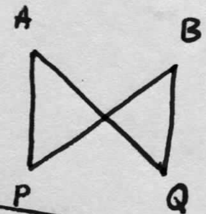
1/19/2018

Hartshorne p.339

Cross/Ratio

better (A, B, P, Q)

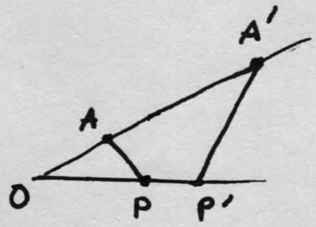
A, B, P, Q distinct pts $\qquad (AB, PQ) := \dfrac{AP/AQ}{BP/BQ}$



$$\boxed{\text{Prop 37.6 Circle inversion preserves CR: } (AB, PQ) = (A'B', P'Q')}$$

Pf Given pts A, P ~~and~~ they have inverses A', P'

$$OA \cdot OA' = r^2 = OP \cdot OP' \Rightarrow \frac{OA}{OP} = \frac{OP'}{OA'}$$



Case 1 $\quad O, A, P$ not colinear

my Lemma $\quad \Delta_{OAP} \sim \Delta_{OP'A'}$ and thus $\boxed{\dfrac{AP}{A'P'} = \dfrac{OA}{OP'}}$

Case 2 $\quad O, A, P$ Colinear $\Rightarrow \quad AP = OP - OA$

we get the same result $\qquad A'P' = OA' - OP'$

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{c-a}{d-b}$$

?

?

we know $\dfrac{OA}{OP} = \dfrac{OP'}{OA'} \cdot \dfrac{OP}{OP'}$

re-arranging $\quad \dfrac{OA}{OP'} = \dfrac{OP}{OA'}$

$$= \frac{OP - OA}{OA' - OP'} = \frac{AP}{A'P'} \square$$

Now if Q is another pt,

Similiarly $\quad \dfrac{AQ}{A'Q'} = \dfrac{OA}{OQ'}$

Divide 'em :

$$\frac{\frac{AP}{A'P'}}{\frac{AQ}{A'Q'}} = \frac{\frac{\cancel{OA}}{OP'}}{\frac{\cancel{OA}}{OQ'}} = \frac{OQ'}{OP'}$$

Now let B be another pt. Working with P and Q as before

$$\frac{\frac{BP}{B'P'}}{\frac{BQ}{B'Q'}} = \frac{\frac{OB}{OP'}}{\frac{OB}{OQ'}} = \frac{OQ'}{OP'} \Rightarrow \frac{\frac{AP}{A'P'}}{\frac{AQ}{A'Q'}} = \frac{\frac{BP}{B'P'}}{\frac{BQ}{B'Q'}}$$

$$\cancel{\frac{A'Q'}{A'Q'}}\ \frac{AP}{\cancel{A'P'}}\ \frac{\cancel{A'Q'}}{AQ} = \frac{BP}{B'P'} \cdot \frac{B'Q'}{BQ}\ \frac{A'P'}{A'Q'} \Rightarrow \frac{AP}{AQ} \cdot \frac{BQ}{BP} = \frac{\cancel{BP}}{B'P'}\ \frac{B'Q'}{\cancel{BQ}}\ \frac{A'P'}{A'Q'} \cdot \left(\frac{\cancel{BQ}}{\cancel{BP}}\right)$$

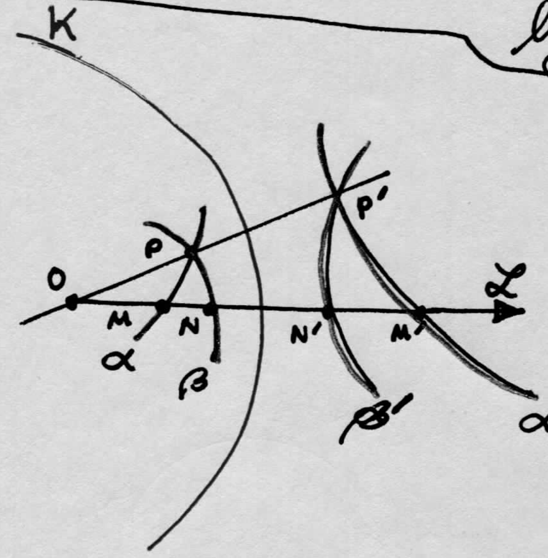
$$\boxed{\frac{AP/AQ}{BP/BQ} = \frac{A'P'/A'Q'}{B'P'/B'Q'}}$$

QED

uses concepts from Calculus and Diff Geo

Kozai & Lob p.13-14

Conformal mapping Thm The magnitude of the angle between 2 intersecting curves is not changed by inversion.



Pf

Inversion circle K has O as its center

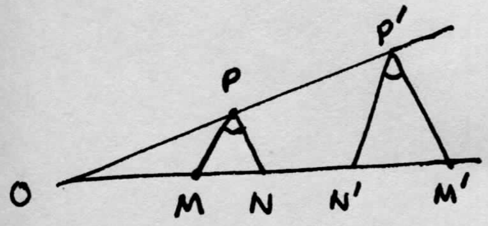
Consider 2 smooth curves α and β which intersect at pt P

These reflect to curves α', β' which intersect at pt P'

Take a ray $\mathcal{L}$ from O that is only a small angle away from $\overrightarrow{OP}$. Then since α, β are smooth and we assume that they cross $\overrightarrow{OP}$ transversally, they also cross $\mathcal{L}$ at pts M, N and thus so do α', β' at N', M'. since $\mathcal{L}$ is preserved by reflection.

we know from the $\boxed{\text{Inversion to Similiar Triangles Lemma}}$

$$\left.\begin{aligned} \measuredangle_{OMP} &= \measuredangle_{OP'M'} \\ \measuredangle_{ONP} &= \measuredangle_{OP'N'} \end{aligned}\right\} \circledast$$



we want to show $\measuredangle_{MPN} = \measuredangle_{M'P'N'}$

$\measuredangle_{OMP}$ is an exterior angle to Δ_{MPN} thus

$\measuredangle_{OMP} = \measuredangle_{MPN} + \measuredangle_{PNM}$ Sum of remote interiors angs.

$$\Rightarrow \measuredangle_{MPN} = \measuredangle_{OMP} - \underbrace{\measuredangle_{MNP}}_{= \measuredangle_{ONP}}$$

same

since O on ray $\overrightarrow{NM}$

$$= \measuredangle_{OMP} - \measuredangle_{ONP}$$

$$= \measuredangle_{OP'M'} - \measuredangle_{OP'N'} \quad \text{by } \circledast$$

$$= \measuredangle_{M'P'N'} \quad \checkmark$$

see picture

▷ Now as the angle between $\overrightarrow{OP}$ and ray $\mathcal{L}$ tends to 0, $M \to P$, $N \to P$ and $M' \to P'$, $N' \to P'$. Segments $\overline{MP}$ and $\overline{NP}$ tend to tangent vectors $\vec{\alpha}$ and $\vec{\beta}$ and similiarly for $\overline{M'P'}$ and $\overline{N'P'}$.

Since $\measuredangle_{MPN} = \measuredangle_{M'P'N'}$ for any non-zero small angle between $\mathcal{L}$ and $\overrightarrow{OP}$, it also holds in the limit, by our assumption of smoothness.

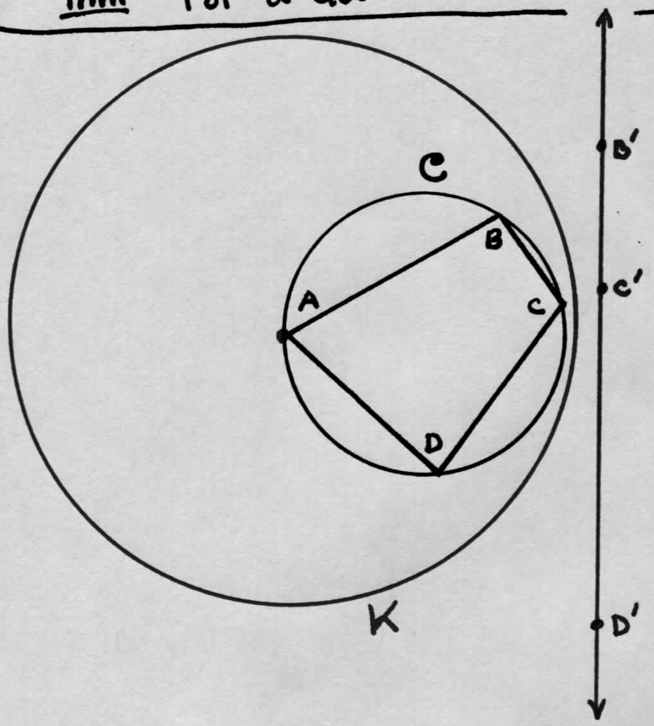
Thus the angle between the 2 curves and 2 image curves is the same. $\boxed{\text{QED}}$

From pdf 'Inversion in a Circle' by Tom Davis www.geometer.org/mathcircles

7/15/2018

Ptolemy's Thm by Inversion

Thm For a Quad inscribed in a circle $C \Rightarrow |AC| \cdot |BD| = |AB||CD| + |BC||AD|$



pf.

Step 1 Draw big circle K centered at vertex A of Quad and make it big enough to enclose circle C.

We know C inverts to a line $\mathcal{L}$ and A is mapped to pt at ∞.

$$\left.\begin{aligned} |AB| \cdot |AB'| &= r^2 \\ |AC| \cdot |AC'| &= r^2 \\ |AD| \cdot |AD'| &= r^2 \end{aligned}\right\} \text{inversion relations}$$

Step 2 By the Key Inversion Lemma

$\Delta_{ABC} \sim \Delta_{AC'B'}$ and

$\Delta_{ADC} \sim \Delta_{AC'D'}$ and $\Delta_{ABD} \sim \Delta_{AD'B'}$

Step 3 By corresp parts of similiar triangs,

- $\dfrac{|B'C'|}{|BC|} = \dfrac{|AB'|}{|AC|} \Rightarrow |B'C'| = \dfrac{|AB'|\,|BC|}{|AC|} \overset{\text{inversion relations 1}}{=} \dfrac{r^2\,|BC|}{|AB|\,|AC|}$
- $\dfrac{|B'D'|}{|BD|} = \dfrac{|AB'|}{|AD|} \Rightarrow |B'D'| = \dfrac{r^2\,|BD|}{|AB|\,|AD|}$
- $\dfrac{|C'D'|}{|DC|} = \dfrac{|AD'|}{|AC|} \Rightarrow |C'D'| = \dfrac{r^2\,|DC|}{|AD|\,|AC|}$

Step 4 B', C', D' all lie on line $\mathcal{L}$ and $|B'D'| = |B'C'| + |C'D'|$

Subs in from <u>step 3</u>:

$$|B'D'| = |B'C'| + |C'D'|$$

$$\frac{\cancel{r^2}\,|BD|}{|AB|\,|AD|} = \frac{\cancel{r^2}\,|BC|}{|AB|\,|AC|} + \frac{\cancel{r^2}\,|DC|}{|AD|\,|AC|}$$

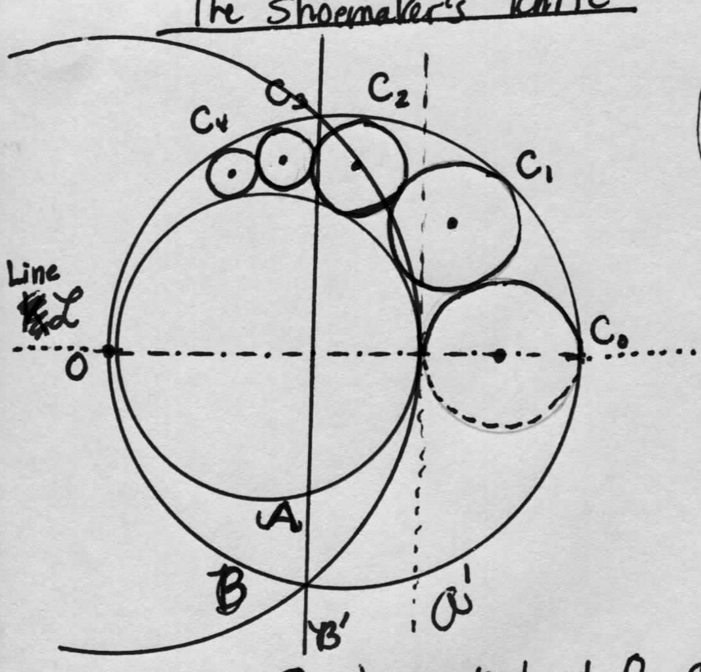
mult both sides by $|AB| \cdot |AC| \cdot |AD|$

$$\Rightarrow |AC|\,|BD| = |AD|\,|BC| + |AB|\,|DC|$$

$\boxed{\text{QED}}$

p.18-19 3/2/2018

The shoemaker's knife



Let d_n = diam of circle C_n

Thm The center of C_n is a distance $n d_n$ from $\mathcal{L}$.

Consider circles $\mathcal{A}$, $\mathcal{B}$, C_0 which all have their centers on line $\mathcal{L}$ and $\mathcal{A}$&$\mathcal{B}$ tangent at O. C_0 is tangent to both $\mathcal{A}$ & $\mathcal{B}$, as is the chain of tangent circles $C_1, C_2, C_3, \ldots$

Let C be centered at O and cut C_2 O.G. [Thm sheet 6]

Invert everything thru C. C_2 is invariant.

$\mathcal{A}$ & $\mathcal{B}$ go thru centre O, so they are mapped to lines and the line is $\perp$ to the diam line of circle, in this case $\mathcal{L}$ Thm sheet 2

~~claim: Image line is parallel to the tangent of circle at O.~~

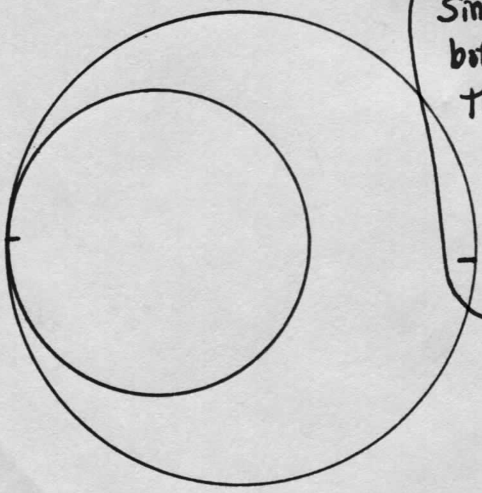
~~so $\mathcal{A}'$, $\mathcal{B}'$ are lines $\perp$ to K.~~

Since $\mathcal{A}$, $\mathcal{B}$ tangent to C_2, so $\mathcal{A}'$, $\mathcal{B}'$ tangent to C_2 as well.

Since circles $\mathcal{A}$ & $\mathcal{B}$ both meet C_2 at 1 pt (tangent) the image lines $\mathcal{A}'$ and $\mathcal{B}'$ also must meet C_2 at one pt.

Thus C_2 is enclosed by 2 tangent vertical lines.

Tangency is preserved because angles are preserved.



~~diam~~ Now consider C_1 which maps to C_1'. C_1' must also be tangent to both $\mathcal{A}'$ and $\mathcal{B}'$, so it must be enclosed by the vertical lines just like C_2.

we know C_1' must be a circle, and the angle between C_1' and $\mathcal{A}'$ must be 0° (conformal map) same for C_1' and $\mathcal{B}'$ $\Rightarrow$ diam C_1' is dist from $\mathcal{A}'$ to $\mathcal{B}'$. C_1' also tangent to C_2.

This arg applies to each C_n.

Hence the result of the inversion is a sequence of circles, all congruent to C_2, and the center of C_0' is on line $\mathcal{L}$

QED