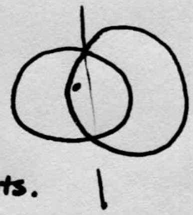
Summary

- Def of Inversion wrt circle
- construction w/ compass & straight edge
- Key Inversion Lemma $\Delta_{OQP} \sim \Delta_{OP'Q'}$
- Thm a line not passing thru O ⟷ circle passing "thru" O
- Line thru O inverts to itself.
- Thm a Circle inside K inverts to a circle outside and vice versa
- Thm If circle $\mathbb{C}$ meets inversion circle K Orthogonally $\Rightarrow \mathbb{C}' = \mathbb{C}$ as a set, not ptwise.



- Any pair of pts lying on same ray from O are inverse pts.

- Thm (Converse) If a circle $\mathbb{C}$ contains a single pair of inverse pts $\Rightarrow$ $\mathbb{C}$ meets K O.G.
- Thm $\mathbb{C}$ is a circle that does not enclose O $\Rightarrow$ $\phi_K\big|_{\mathbb{C}}$ is a dilation $\Rightarrow \mathbb{C}'$ is again a circle

- Co-ord expressions for inversion

- Peaucellier Linkage

- Circle Inversion preserves cross-ratio
- Conformal mappings – Inversion preserves angles
- Ptolemy's Thm proved by Inversion
- The shoemaker's knife Arbelos

Inversion in a Circle

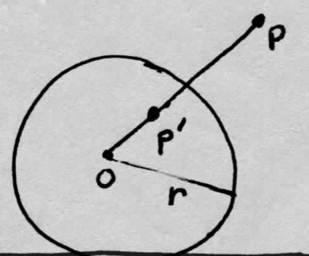
1/2/2018

Well, its conformal ①

Inverse of Pt wrt Circle (or 'Reflection' thru Circle)

Euclidean

↳ does ~~not~~ preserve dist and ~~angles~~



a pt P outside a circle $K(o,r)$ has an inverse pt inside K that lies on the radial line $\overline{OP}$. ~~the inverse is~~ P' where

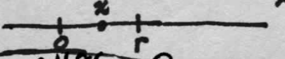
$|OP| \cdot |OP'| = r^2$.

Likewise P is inverse of P'

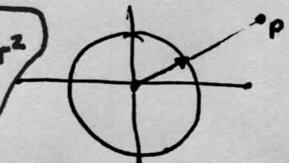
REWRITE THIS AND PUT A SUMMARY OF THIS SECTION

The prototype is the map $x \mapsto \frac{1}{x}$ on the Real line

To generalize this property, we must take $x \longmapsto \frac{r^2}{x}$ so $x\left(\frac{r^2}{x}\right) = r^2$



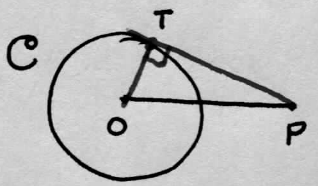
To see this, think of vectors in $\mathbb{R}^2$



Given radius vector $\vec{r}$, $\vec{p} = \alpha\vec{r}$ $\quad \alpha > 1$

$\vec{p}' = \frac{1}{\alpha}\vec{r}$ and $\vec{p}\cdot\vec{p}' = \frac{1}{\alpha}\vec{r}\cdot\alpha\vec{r} = \|r\|^2$.

p.486 Construct Inverse of P with Straight-edge and compass



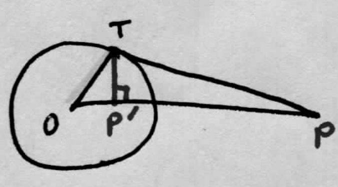
Given O and P, ① construct tangent line $\overline{PT}$ to circle as per sheet (35)

It meets C at one pt T

Then $OT \perp PT$ by

② ~~Draw perp~~ Drop a line seg down from T that is perp to $\overline{OP}$ (thru a pt not on line ∃! perp)

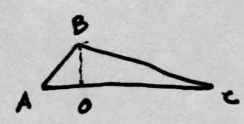
This pt P' is the inverse we seek



How do we know this? Because

Δ_{OPT} is Right Δ with hypot $\overline{OP}$

$\overline{P'T}$ is altitude to hypotenuse



Alt wrt hypot thm 61 sheet (21) COR 2 $\Rightarrow \dfrac{|OP|}{|OT|} = \dfrac{|OT|}{|OP'|}$

Each leg is geo mean between hypot and O.G. proj of leg onto hypot

but $|OT| = r \quad \Rightarrow \dfrac{|OP|}{r} = \dfrac{r}{|OP'|}$

$\Rightarrow |OP| \cdot |OP'| = r^2$ $\boxed{\text{QED}}$

Reverse the steps to find P, given P'.

Let us denote inversion wrt circle C as $\phi_C : E - \{0\} \longrightarrow E - \{0\}$

$\phi_C^2 = Id$

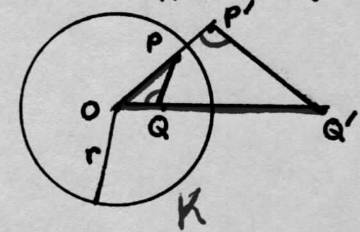
Our familiar reflection across a line is a special case of circle inversion when the radius becomes infinite.

√

Inversion to similiar triangles

meaning O-P-Q is not on a line.

Lemma 2 non-colinear pts form similiar triangles with the center of circle under inversion:

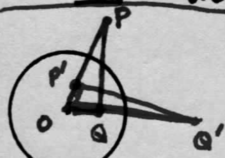


$$\Delta_{OPQ} \sim \Delta_{OQ'P'}$$

specifically $OP \to OQ'$, $OQ \to OP'$, $PQ \to Q'P'$

COR $\measuredangle_{OQP} = \measuredangle_{OP'Q'}$

Pf. we know $|OP||OP'| = r^2 = |OQ||OQ'|$



Proof still works even in a case like this.

$$\frac{OP \cdot OP'}{OQ'} = \frac{r^2}{OQ'} = OQ$$

SAS Similiarity

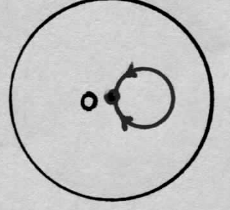
$$\frac{OP}{OQ'} = \frac{r^2}{OP'\,OQ'} = \frac{OQ}{OP'} \Rightarrow \begin{matrix} OP = \lambda OQ' \\ OQ = \lambda OP' \end{matrix} \Longrightarrow \Delta_{OPQ} \sim \Delta_{OQ'P'}$$

similiarity scale factor λ $\qquad \measuredangle_0 = \measuredangle_0$ in both Δ

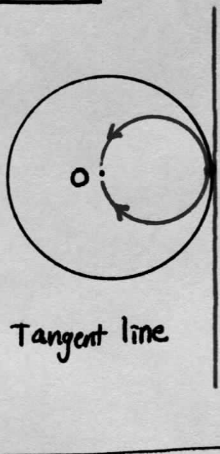
Corresponding angles $\Rightarrow \measuredangle_{OQP} = \measuredangle_{OP'Q'}$

□

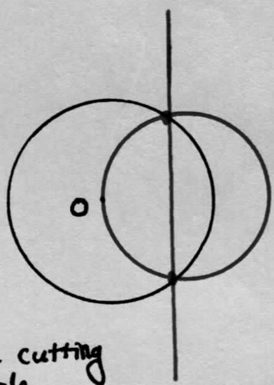
Reflections of Lines and Circles



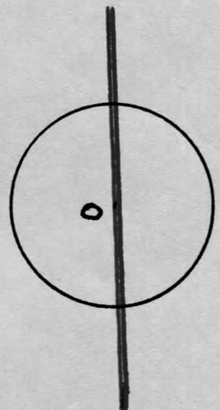
Distant line ⟷ small circle going thru center. Center pt is the 'pt at ∞' so we could say it is a punctured circle.



Tangent line



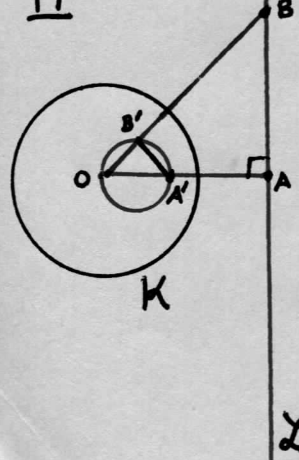
Line cutting circle.



Line thru center maps to itself — circle with infinite radius

Thm A Line not passing thru center O inverts to a Circle passing thru O — and vice versa. [well, O maps to 'pt at ∞'] so could say 'punctured circle'

Pf



∃! a perp seg from line $\mathcal{L}$ going thru O; call it $\overline{OA}$

The pt A has a reflection A' where $OA \cdot OA' = r^2$

choose another pt B on $\mathcal{L}$, it reflects to B'

B' cannot be on line $\overline{OA'A}$ so we can apply the $\boxed{\text{Lemma}}$

$$\Delta_{OB'A'} \sim \Delta_{OAB} \text{ and } \measuredangle_{OB'A'} = \measuredangle_{OAB} = 90° \text{ since } \perp$$

By the Converse to Inscribed Angles Thm (specifically conv to Thales' Diameter thm), B' lies on a half circle $C_{\mathcal{L}'}$ (with OA' as a diameter) either above or below $\overline{OA'}$.

Since B is arb, the whole circle is filled in, except for pt O, which corresponds to B at ∞.

see next page

cont'd →

Cont'd <u>Onto</u>

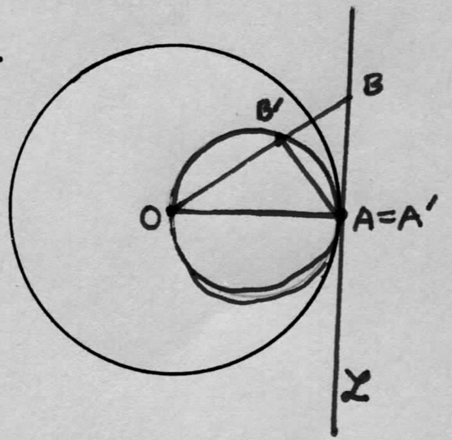
Is there any pt X on the circle with diam $\overline{OA'}$ that is <u>not</u> the image of some $B \in \mathcal{L}$?

Choose any pt X. Reflect it to X'. Is $X' \in \mathcal{L}$?

We know $\Delta_{OXA'} \sim \Delta_{OAX'}$ and $\underbrace{\measuredangle_{OXA'}}_{90^\circ \text{ by Thales' Diam Thm}} = \measuredangle_{OAX'}$

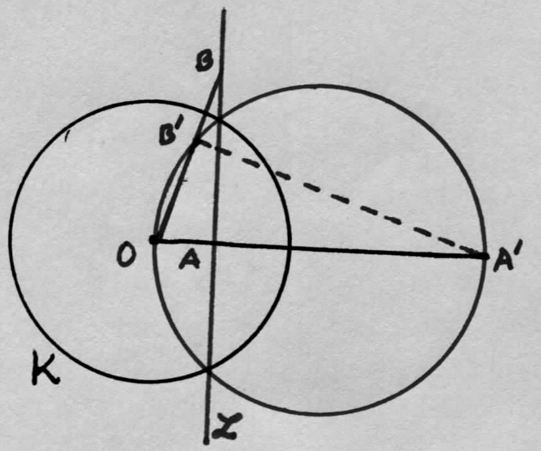
So $\measuredangle_{OAX'} = 90^\circ$ and since there is only 1 line thru pt A that is $\perp$ to $\overline{OA}$, $X' \in \mathcal{L}$ $\square$

<u>Case ii</u>



This case works just like the previous one.

<u>Case iii</u>



If B is inside inversion circle K, then just apply the Lemma

$\Delta_{OAB} \sim \Delta_{OB'A'}$ and $\underbrace{\measuredangle_{OAB}}_{90^\circ} = \measuredangle_{OB'A'}$

$\overline{OA'}$ is still diam of image circle

Since $\measuredangle_{OB'A'} = 90^\circ$ we apply Converse of Thales' Diam Thm just like before.

If B outside K (and A is inside)

we can <u>still</u> apply the Lemma

$\Delta_{OAB} \sim \Delta_{OB'A'}$

$\underbrace{\measuredangle_{OAB}}_{90^\circ} = \measuredangle_{OB'A'}$

$\overline{OA'}$ is still the diam, apply Conv. to Thales' D Thm just like before

$\boxed{\text{QED}}$

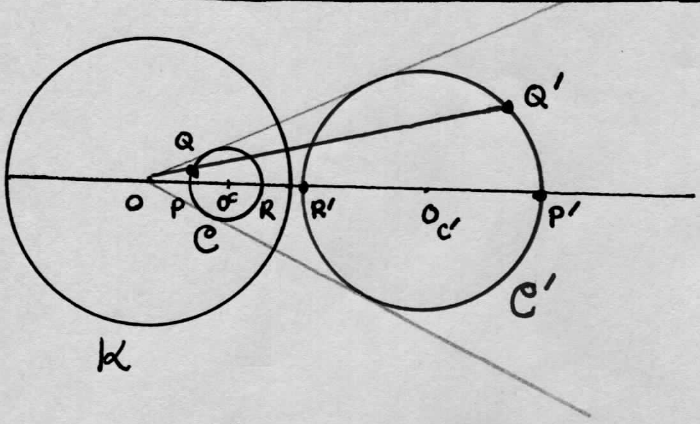
Pdf off web: 'Circle Inversions and Applications to Euclidean Geo' Kozai & Libeskind

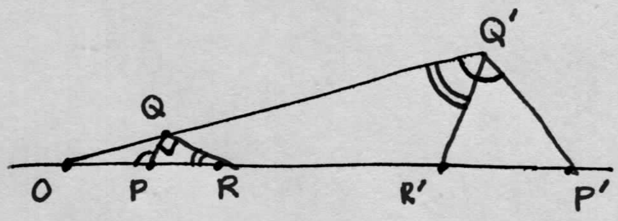
1/15/2018

(4')

Thm The reflection [redacted] thru K of a circle C inside K not passing thru O is another circle C' outside K, and vice versa.

Pf





Circle C has center O_c and pts P, R are the endpts of its diam along ray $\overrightarrow{OO_c}$ (P is closest to O). also on $\overrightarrow{OO_c}$, outside K, are reflection pts R', P' (P' furthest from O).

NOTE: center does not map to center $O_c \not\mapsto O_{c'}$ see p.

We denote the image of C as C'. Is C' a circle?

Let Q be any pt on C (not P or R). From Thales' Diam thm, we know $\measuredangle_{PQR} = 90°$

If we could show $\measuredangle_{P'Q'R'} = 90°$ then by Conv to Thales D thm, we would know that all pts Q' lie on a circle. [Then there is still the question if $\sigma(C)$ fills the circle (maps onto)]

$$OP \cdot OP' = OQ \cdot OQ' = OR \cdot OR' = r^2$$

Want to show $90° = \measuredangle_{PQR} = \measuredangle_{P'Q'R'}$

By Lemma, $\Delta_{OPQ} \sim \Delta_{OQ'P'} \Rightarrow \measuredangle_{OPQ} = \measuredangle_{OQ'P'}$ (★)

By Lemma, $\Delta_{ORQ} \sim \Delta_{OQ'R'} \Rightarrow \measuredangle_{ORQ} = \measuredangle_{OQ'R'}$ (★★)

By (★) $\quad \measuredangle_{OPQ} = \measuredangle_{OQ'P'}$

It is an exterior angle to Δ_{PQR}

$\measuredangle_{PRQ} + \measuredangle_{PQR} = \measuredangle_{OQ'R'} + \measuredangle_{P'Q'R'}$ since segment $R'Q'$ cuts the big angle.

$\measuredangle_{PRQ} + \measuredangle_{PQR} = \measuredangle_{ORQ} + \measuredangle_{P'Q'R'}$ by (★★)

$\cancel{\measuredangle_{ORQ}} + \measuredangle_{PQR} = \cancel{\measuredangle_{ORQ}} + \measuredangle_{P'Q'R'}$ since O is on same line seg as P, the angles $\measuredangle_{PRQ} = \measuredangle_{ORQ}$

$\Rightarrow \quad \measuredangle_{PQR} = \measuredangle_{P'Q'R'}$

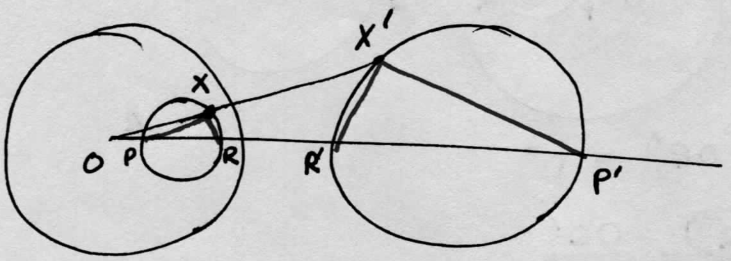
We know $\measuredangle_{PQR} = 90$ by Thales Diam thm, since C is a circle. Thus $\measuredangle_{P'Q'R'} = 90°$ and all pts Q' lie on a circle. $\Rightarrow C'$ is contained in a circle. Is it all of it?

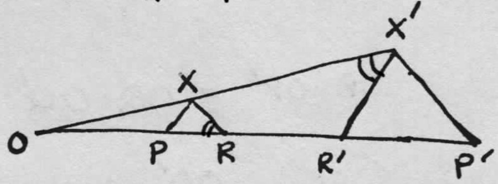
cont'd →

What about an arb pt $X' \in$ Circle with diam $\overline{R'P'}$?

We know then by Thales D

$$\measuredangle_{R'X'P'} = 90$$





It plainly has image X inside K. Is X on C?

It would be if $\measuredangle_{PXR} = 90$

Play the same game as before $\Delta_{OPX} \sim \Delta_{OX'P'} \Rightarrow \measuredangle_{OPX} = \measuredangle_{OX'P'}$ ✱

$\Delta_{ORX} \sim \Delta_{OX'R'}$ $\measuredangle_{ORX} = \measuredangle_{OX'R'}$ ✱✱

By ✱ $\measuredangle_{OPX} = \measuredangle_{OX'P'}$

By exactly the same steps as before, we get $\measuredangle_{PXR} = \measuredangle_{P'X'R'}$ but this time it is the RHS that is known to equal $90°$.

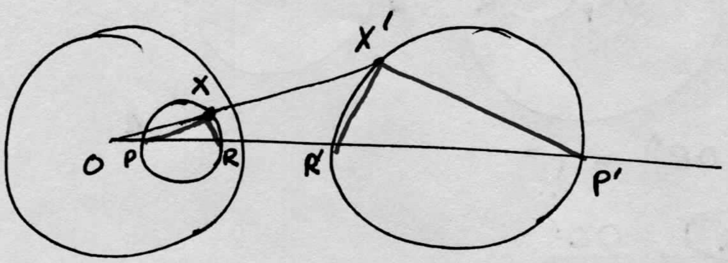
QED

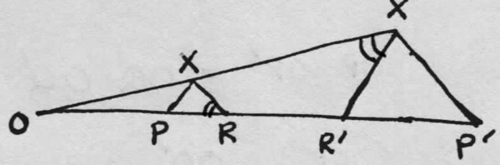
$$\boxed{\text{What if } C \text{ intersects } K?}$$

What about an arb pt $X' \in$ Circle with diam $\overline{R'P'}$?

We know then by Thales D

$\measuredangle_{R'X'P'} = 90$





It plainly has image X inside K. Is X on C?

It would be if $\measuredangle_{PXR} = 90$

Play the same game as before $\Delta_{OPX} \sim \Delta_{OX'P'} \Rightarrow \measuredangle_{OPX} = \measuredangle_{OX'P'}$ (*)

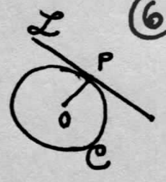
$\Delta_{ORX} \sim \Delta_{OX'R'}$ $\measuredangle_{ORX} = \measuredangle_{OX'R'}$ (**)

By (*) $\measuredangle_{OPX} = \measuredangle_{OX'P'}$

By exactly the same steps as before, we get $\measuredangle_{PXR} = \measuredangle_{P'X'R'}$ but this time it is the RHS that is known to equal 90°.

QED

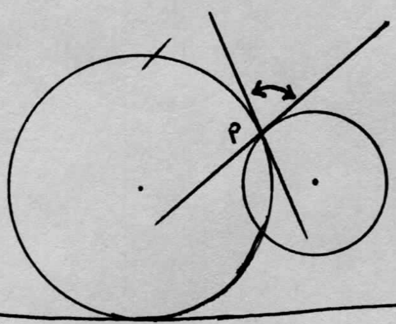
$\boxed{\text{What if C intersects } K\text{?}}$



Thm at any given pt P on a circle C, $\exists!$ tan line to C.

pf. Tan line $\mathcal{L}$ must be $\perp$ to radial seg $\overline{OP}$
By $\boxed{\text{Thm 23 sheet 14}}$ $\exists!$ a line thru P $\perp$ to $\overline{OP}$

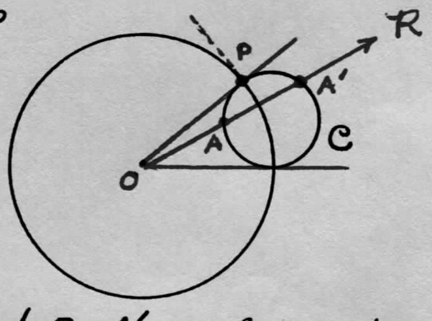
Def If 2 circles meet at a pt P, the angle between them is defined to be the angle between their unique tan lines at P (the one $\leq 90°$)



Hartshorne G:EAB
P. 336

Thm If circle C meets inversion circle K O.G. $\Rightarrow$
- $C' = C$ that is, C is invariant under inversion as a set, but the pts are moved around.
- any pair of C pts lying on the same ray from O are inverse pts

pf ① Radius $\overline{OP}$ is tangent to C because:
we know tangent line $T_K \perp T_C$ at P (Given)
we know $T_K \perp \overleftrightarrow{OP}$
Since both T_C and $\overleftrightarrow{OP}$ are $\perp$ to T_K at P, and perp lines are unique $\boxed{\text{Thm 23 sheet 14}}$ $T_C = \overleftrightarrow{OP}$



② Let ray R from O cut thru C at pts A, B
Its a secant line
By $\boxed{\text{Euclid III.36 (Power of a Pt) sheet 36a}}$ $\underbrace{OP \cdot OP}_{r^2} = OA \cdot OB$
But then A & B satisfy the inversion relation and $B = A'$. $r^2 = OA \cdot OA'$
③ Line R was arb, thus any 2 pts on C cut by the same ray from O are inverses.

Converse Thm If circle C contains a single pair of inverse pts A, A' distinct $\Rightarrow$ C meets K O.G.
$C' = C$ ← This follows from prev thm.

Pf. Since A reflects to A', one pt must be inside K and the other outside.
By continuity, C must meet K in 2 pts — call one P
Draw radial seg $\overline{OP}$ $|OP| = r$
A, A' inv $\Rightarrow$ $OA \cdot OA' = r^2 = OP \cdot OP$
By Euclid III.37 (sheet 36a) $\overline{OP}$ is actually a tangent line to C at P
Do we have tan line $T_K \perp T_C$? Yes, since the tan line at P to C is unique and $\overline{OP} \perp T_K$
What about the other pt where C meets K, call it Q.
we can apply the exact same arg

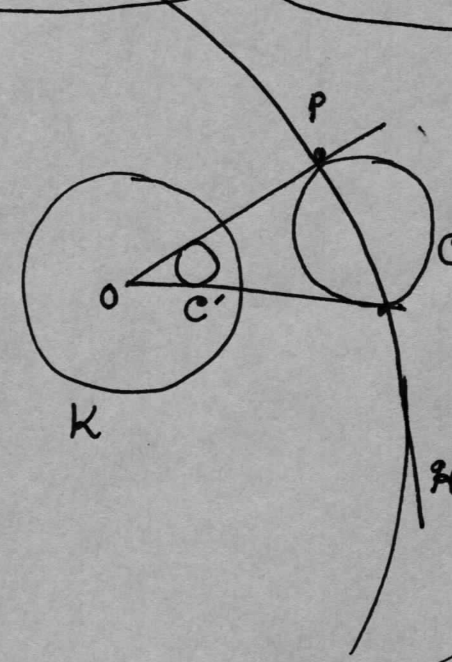
$\square$

Hartshore p.331 reworked to be clearer

1/26/2018 7

$\mathcal{C}$ is a circle that does not enclose O. Here it is drawn outside K, but we could just as easily have started with $\mathcal{C}'$.

Show $\mathcal{I}_K\big|_{\mathcal{C}}$ is a dilation thus $\mathcal{C}'$ is again a circle.



pf. Because $\mathcal{C}$ does not encircle O, we can draw tan line $\overline{OP}$

Draw big circle $\mathcal{H}$ from O with radius $|OP|$

By construction, $\mathcal{H}$ is O.G. to $\mathcal{C}$ so

$$\mathcal{I}_{\mathcal{H}}(\mathcal{C}) = \mathcal{C} \quad \circledast$$

Define $\theta \cdot \mathcal{I}_K \circ \mathcal{I}_{\mathcal{H}} : \mathbb{E} \to \mathbb{E}$

By the lemma, this is a dilation of the plane $\mathbb{E}$

But $\underbrace{\theta(\mathcal{C})}_{\text{circle, since it is a dilation.}} = \mathcal{I}_K(\mathcal{C})$ by $\circledast$ This only holds for $\mathcal{C}$, not some other set.

Case 2 $\mathcal{C}$ encircles O. I couldn't figure this out, but the other pf should cover this case (and this pf covers the case where $\mathcal{C}$ intersects K ~~but~~ even not O.G. ~~g~~)

Then we can't form tangent line $\overline{OP}$.

$\triangleright$ Is the center of $\mathcal{C}$ mapped to center of $\mathcal{C}'$. No!

Lemma: • The composition of 2 circle inversions wrt the same center O is a dilation or contraction of the plane [a similarity transform w/ const scale factor]

• This preserves any shape, but not its size

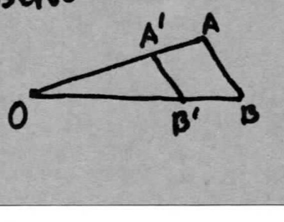
pf Since the inversions have the same center O, it is enough to consider any radial line, which we take as the x axis.

Let $\mathcal{I}_1 : x \longmapsto \frac{r^2}{x}$ and $\mathcal{I}_2 : x \longrightarrow \frac{R^2}{x}$

Then $\mathcal{I}_2 \circ \mathcal{I}_1(x) = \frac{R^2}{(r^2/x)} = \left(\frac{R^2}{r^2}\right)x$ and the scale factor $\lambda = \frac{R^2}{r^2}$ is the same for any radial line.

This ~~m~~ composite map leaves O fixed. ▨

Observe



These triangles are similiar $\Delta_{OAB} \sim \Delta_{OA'B'}$ but this really doesn't tell us anything.