5/27/2017

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Convexity

a **set** K is convex if for any $x, y \in K$, the line segment between x and y is also in K i.e. $\boxed{z = (1-\lambda)x + \lambda y}$ $z \in K \ \forall \lambda \in [0,1]$

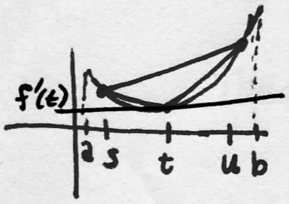
a **fcn** is convex if (1) Its domain is a convex set

"sub linear"? → (2) $f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y)$

(b) **Strictly** convex: $f((1-\lambda)x + \lambda y) < (1-\lambda)f(x) + \lambda f(y)$

(c) **Strongly** convex:

Rudin RACA p.60-61



$f : (a,b) \longrightarrow \mathbb{R}$

The locus of pts $\langle (1-\lambda)x + \lambda y, (1-\lambda)f(x) + \lambda f(y) \rangle$ defines the graph of a line segment connecting $\langle x, f(x) \rangle$ to $\langle y, f(y) \rangle$ as λ varies $0 \le \lambda \le 1$

a norm $\|\cdot\|$ is never **strictly** convex because we get equality if one vector is a scalar mult of the other.

If $y = kx$ then $\|(1-\lambda)x + \lambda y\| = \|(1-\lambda)x + \lambda k x\|$

$= \|[1-\lambda + \lambda k)x\| = (1-\lambda + \lambda|k|)\|x\|$

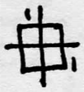
$= (1-\lambda)\|x\| + \lambda|k|\|x\|$

$= (1-\lambda)\|x\| + \lambda\|y\|$

But a norm is called "strictly convex" if its unit ball contains every line segment in its interior.

Thus $\|x\|_p = (\Sigma x_i^p)^{1/p}$ is

but $\|x\|_\infty = \max_i |x_i|$ is not



strict convexity means line segment lies above graph of f (Regular convexity allows it to coincide).

▷ Let's show $\dfrac{f(t) - f(s)}{t - s} \le \dfrac{f(u) - f(t)}{u - t}$

Key: let $t = (1-\lambda_t)s + \lambda_t u$

$$f(t) = f((1-\lambda_t)s + \lambda_t u) \le (1-\lambda_t)f(s) + \lambda_t f(u)$$

Just plug it in:

$$\frac{f(u) - f(t)}{u-t} \ge \frac{f(u) - [(1-\lambda_t)f(s) + \lambda_t f(u)]}{u - [(1-\lambda_t)s + \lambda_t u]} = \frac{(\cancel{1-\lambda})f(u) - (\cancel{1-\lambda})f(s)}{(\cancel{1-\lambda})u - (\cancel{1-\lambda})s} = \frac{f(u) - f(s)}{u - s}$$

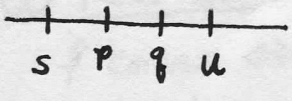
and

$$\frac{f(t) - f(s)}{t - s} \le \frac{(1-\lambda)f(s) + \lambda f(u) - f(s)}{(1-\lambda)s + \lambda u - s} = \frac{\cancel{\lambda}(f(u) - f(s))}{\cancel{\lambda}(u - s)}$$

same

combine them: $\dfrac{f(t) - f(s)}{t - s} \le \dfrac{f(u) - f(s)}{u - s} \le \dfrac{f(u) - f(t)}{u - t}$ QED

▷ Now show $f'(s) \le f'(u)$ if f difb.



$$\frac{f(p) - f(s)}{p - s} \le \frac{f(u) - f(q)}{u - q}$$

$$\lim_{p \searrow s} f'(s) \le \frac{f(u) - f(q)}{u - q}$$

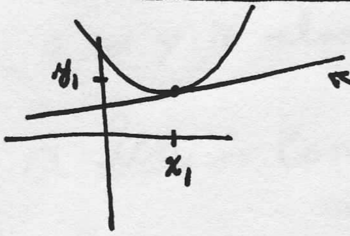
Now take lim on both sides of eq: $\lim_{q \nearrow u}$

If f difb, one sided lim is equal to 2 sides

$\Longrightarrow f'(s) \le f'(u)$ □

Cont'd →

Now show graph(f) always lies above the tangent line



what is the eq. of this line? $\frac{y-y_1}{x-x_1} = m \Rightarrow y - y_1 = m(x-x_1)$

$\Rightarrow y = y_1 + m(x-x_1)$

$= f(x_1) + f'(x_1)(x-x_1)$

why is $f(x) \geq y$? i.e. above line?

Because $\frac{y - f(x_1)}{x - x_1} = f'(x)$ BUT for any $\xi > x_1$, $f'(x_1) \leq f'(\xi)$

and by MVT $\frac{f(x)-f(x_1)}{x-x_1} = f'(\xi)$ for $\xi \in (x_1, x)$

Thus $f'(x_1) \leq f'(\xi)$ i.e. $f'(x_1) \leq \frac{f(x)-f(x_1)}{x-x_1}$

$\Rightarrow \boxed{f(x_1) + f'(x_1)(x-x_1) \leq f(x)}$

we still need to consider $x < x_1$:

$\frac{f(x_1)-f(x)}{x_1-x} \overset{MVT}{=} f'(\eta) \leq f'(x_1)$ or $f(x_1) - f(x) \leq f'(x_1)(x_1 - x)$

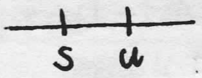
mult both sides by (-1):

$f(x) - f(x_1) \geq f'(x_1)(x - x_1)$

$\Rightarrow \boxed{f(x) \geq f(x_1) + f'(x_1)(x-x_1)}$

SAME

Now show $f'' > 0 \Rightarrow f$ convex



we know $f'(s) \leq f'(u) \Rightarrow 0 \leq f'(u) - f'(s)$

$0 \leq \frac{f'(u) - f'(s)}{u - s}$

Take $\lim_{s \nearrow u}$ $0 \leq f''(u)$ □

Notes from Wikipedia:

f convex $\Rightarrow f$ cont   see RACA ch 3 prb 3

f convex, difb $\Rightarrow f$ C^1

f convex: "f is incr at an incr rate"

If $f \in C^2$ we can use the following tests:

f convex $\Leftrightarrow f''(x) \geq 0 \;\forall x$

f strictly convex $\Rightarrow f''(x) > 0$ [no reverse implication $\Leftarrow$, $f(x) = x^4$ is strictly convex but $f''(x) = 12x^2$ and $f''(0) = 0$]

f strongly convex $\Leftrightarrow f''(x) \geq m > 0$

e.g. $f(x) = x^2$ has $f''(x) = 2 > 0$ on $\mathbb{R}$

$f(x) = e^x$ has $f''(x) = e^x$ but can come arb close to 0 as $x \searrow -\infty$

□

Jensen's Ineq f convex

$E[f(X)] \geq f(E[X])$

Examples: $f(x) = |x|$ convex

$f(x) = x^2$ strongly convex

$f(x) = e^x$ strictly

This is the strongest.