Summary of ODEs Mostly Following Boyce & di Prima Ch 2

8/17/2015

Consider just the plane $\mathbb{R}^2$. Let y be a fcn of x.

The most general ODE would be $F(x, y, y', \dots, y^{(n)}) = 0$

we shall simplify down to the case $\boxed{y' = f(x,y)}$

but first a little more of the general discussion:

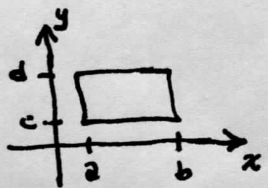
<u>Explicit soln</u>: $y = \varphi(x)$ satisfies $y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$ $\forall x \in (\alpha, \beta)$.

<u>Implicit soln</u>: $\Psi(x,y) = 0$ implicitly defines φ that solves above eq.
(These solns have problems with domain of definition, etc...)

p.24 <u>Thm 2.2</u>

Cauchy IVP

Let $R := [a,b] \times [c,d]$ (simply conn) rectangle in $\mathbb{R}^2$
$f: R \to \mathbb{R}$ cont
$D_2 f$ cont
$y' = f(x,y)$ on R
pt $(x_0, y_0) \in R$



$\Rightarrow$ Given the IVP $y' = f(x,y)$, $y(x_0) = y_0$

$\exists !$ soln fcn y which is dif'b in some interval about x_0: $[x_0 - h, x_0 + h]$

If f is <u>linear</u> wrt y, we get y is defined on whole interval $[a, b]$.

<u>pf</u> see Cheney Applied Math I handout book.

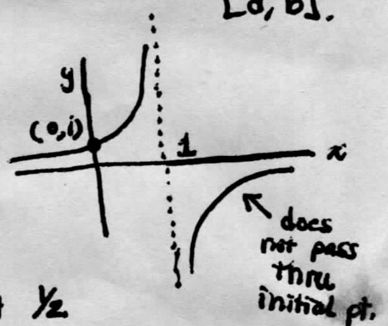
<u>Some counterexamples</u>

1. Bad behaviour of non-linear prob:

Goes thru pt $(0,1) \to$

$y' = y^2$
$y(0) = 1$

By direct subs, $y(x) = \frac{1}{1-x}$ is a soln valid for $(-\infty, 1)$ but no indication from ODE that $x = 1$ should be a singularity.



Now use IC $y(0) = 2$ $\quad y(x) = \frac{2}{1-2x}$ is soln with sing at $\frac{1}{2}$
singularities can move around depending on ICs.

Bernoulli: $y' + p(x) y = q(x) y^n$

2. Relax conds in Cauchy IVP and get <u>non-uniqueness</u>

$y' = y^{1/3}$
pt $(0,0)$ $\quad y(0) = 0$

- $\frac{dy}{dx} = y^{1/3} \Rightarrow \int y^{-1/3} dy = \int dx \Rightarrow \frac{3}{2} y^{2/3} = x + c \Rightarrow y = \left(\frac{2}{3}x\right)^{\pm 3/2}$ This goes thru the pt $(0,0)$
- Also $y(x) \equiv 0$ is a soln that goes thru $(0,0)$.

Why? Because $f(x,y) = y^{1/3}$

$D_2 f = \frac{1}{3} \frac{1}{y^{2/3}}$ This is not cont (or even defined) on line $y = 0$ so it is not at pt $(0,0)$

(For another pt where $y \neq 0$, we would get a unique soln passing thru it).

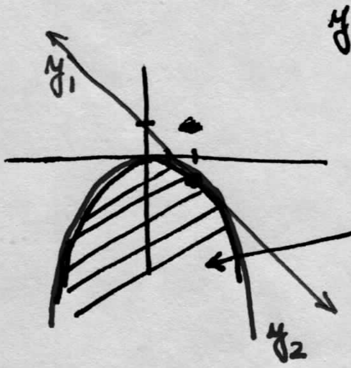
Counterexample 3 ch 2.3 p.28 #6

With non-linear probs, we can't nec get <u>all</u> soln's by varying the arb const.

$$\begin{cases} y' = -\frac{1}{2}x + \frac{1}{2}(x^2+4y)^{1/2} \\ y(2) = -1 \end{cases}$$

$f(x,y)$ is not defined for $x^2+4y<0$

$y < -\frac{1}{4}x^2$



Here are 2 solns: $y_1(x) = 1-x$

$y_2(x) = -\frac{1}{4}x^2$

$y(x) = cx + c^2$ satisfies ODE and if $c=-1$ we also satisfy the IC $y(2)=-1$. But for no value of c does $y = y_2$.

The real Thm is Avez DC Thm 7.1 p.82

(i) open set $\mathcal{U} \subseteq E$ Banach sp.

(ii) vf $X: \mathcal{U} \longrightarrow E$ satisfies

Lipschitz Cond $\|X(x) - X(y)\|_E \leq k\|x-y\|_E$

(iii) $\bar{B}(x_0, r) \subset \mathcal{U}$

(iv) $\sup_{x\in\bar{B}} \|X(x)\|_E \leq M$

$\Rightarrow$ $\exists!$ dif'b soln curve $\varphi_{x_0}: [t_0 - \frac{r}{M}, t_0 + \frac{r}{M}] \longrightarrow \bar{B}(x_0, r)$

that is, φ_{x_0} satisfies

$$\begin{cases} \dot{\varphi}_x = X(\varphi_x(t)) \\ \varphi_{x_0}(t_0) = x_0 \end{cases}$$

So from this we can see that counterexample 3 did not have a ball $B(x_0, r) \subset \mathcal{U}$ for pt $x_0 = \langle 2, -1\rangle$. This pt was on the boundary of $\mathcal{U}$. Note also x plays the role of time t in ex 3, and Avez is talking about autonomous ODEs.

BIG IDEAS

we have $y' = f(x,y)$ i.e. $\boxed{\frac{dy}{dx} = f(x,y)}$ ★

another popular form is

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0} \quad ★★$$

$$\Leftrightarrow M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$\Leftrightarrow \frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)}$$

so given $y' = f(x,y)$, even if f won't factor nicely into a quotient $\frac{M}{N}$, we can always take $M := -f(x,y)$, $N := 1$

Now lets consider the special cases that we can solve:

Separable: $M = M(x)$
$N = N(y)$

Exact: $\exists\ \varphi(x,y) \ni \frac{d}{dx}\varphi = \underbrace{D_1\varphi}_{M} + \underbrace{D_2\varphi}_{N} y'$ That is $M = D_1\varphi$
$N = D_2\varphi$

This is just saying we have a 'Conserved Quantity' or 'First Integral' φ ?

Make ODEs exact by finding integrating factor μ

Linear: $y' + p(x)\,y = q(x)$

we call this linear because define operator

$$L(y) = \frac{d}{dx}(y) + p(x)\,I(y)$$
$$= (D_x + pI)(y)$$

Linear because (α scalar; does not depend on x)

$$L(\alpha f + g) = (\alpha f + g)' + p(\alpha f + g)$$
$$= \alpha f' + g' + p\alpha f + pg$$
$$= \alpha f' + \alpha p f + g' + pg$$
$$= \alpha L(f) + L(g)$$

Homogeneous $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

Bernoulli - out of seq from ch 2.2 p. 24

Ricatti

Clairut

Brief discussion of Lie

Separable eq ex $\frac{dy}{dt} = ky \Rightarrow \int \frac{1}{y}\,dy = \int k\,dt + c_0$

To be justified below

$$\Rightarrow \ln|y| = kt + c_0$$

$$|y(t)| = e^{kt}e^{c_0} = c_1 e^{kt} \quad \text{const } c_1 > 0$$

$$y(t) = \pm c_1 e^{kt}$$

So we write $y(t) = Ce^{kt}$ where C can be pos or neg

This is a soln to the ODE (check by substituting it in) and

C would be determined by IC $y(0) = C$

ex $\begin{cases} \frac{dy}{dx} = \frac{x^2}{1+y^2} \\ y(2) = 1 \end{cases} \Rightarrow \underbrace{-x^2}_{M} + \underbrace{(1+y^2)}_{N}\frac{dy}{dx} = 0 \Rightarrow (1+y^2)\,dy = x^2\,dx$

integrate $\int x^2\,dx + c = \int (1+y^2)\,dy \Rightarrow \frac{1}{3}x^3 + c = y + \frac{1}{3}y^3$

Plug in IC to solve for c: $\frac{1}{3}2^3 + c = 1 + \frac{1}{3}1^3 \Rightarrow c = -4/3$

$\Rightarrow y^3 + 3y - x^3 + 4 = 0$ This is the general kind of soln we get — a conserved quantity, but not an explicit form $y = y(x)$

General Discussion we could have $M(x,y) + N(x,y)\,y'(x) = 0$

But in the special case $M(x) + N(y)\,y' = 0$ we can recognize the LHS as the derivative of something.

Define $g(x)$ and $h(x)$ $\ni$ $g'(x) := M(x)$, $h'(y) := N(y)$ This is always possible $g(x) := \int_a^x M(x)\,dx$

$$\Rightarrow \frac{d}{dx}\left[g(x) + h(y(x))\right] = 0 \qquad g' + h'(y)\,y' = M + Ny' = 0$$

$\Rightarrow \boxed{g(x) + h(y(x)) = c}$ So we have a conserved quantity or const. of motion

[If we could invert h, we could explicitly solve: $y(x) = h^{-1}(c - g(x))$]

ASIDE: Justifying Separation of Variables

How can we say $M(x) + N(y)\frac{dy}{dx} = 0 \Rightarrow M(x)\,dx + N(y)\,dy = 0$?

Really we <u>can't</u> — see discussion on sheet 6a

What we say rigorously is $\boxed{\int N(y)\,dy = -\int M(x)\,dx + c}$

Let's prove it: If $y = y(x)$ on $[a, b]$

cont'd →

we have $N(y(x))\,y'(x) = -M(x)$

Integrate both sides wrt dx : $\int_a^s N(y(x))y'(x)\,dx = \int_a^s -M(x)\,dx$
(s is the variable pt now)

$y = y(s)$ $\qquad \underbrace{\int_a^s N(y(x))y'(x)\,dx}_{\text{COV Thm}} = \int_{y(a)}^{y} N(y)\,dy$

Then we could just not write the limits of integration and instead put a const term:

$$\int N(y)\,dy = \int -M(x)\,dx + C \qquad \square$$

▷ For completeness, let's also give a proof of COV Thm:

$$\boxed{\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du}$$

'unpacked' form $\qquad$ 'packed up' form

cf Ross EA:TOC
pp.201
also my Swokowski
ch 9 writeup

Pf $\quad$ Step 1 $\quad$ we want $f(g(x))\,g'(x) = p'(x)$ for some p

Define $p(x) := F(g(x))$ where $F(s) := \int_c^s f(t)\,dt$ for some fixed $c \in (a,b)$

Then $p'(x) = F'(g(x))\,g'(x)$

and by FTOC(I) (diff an int) $F'(s) = f(s)$ so $F'(g(x)) = f(g(x))$

$\Rightarrow p'(x) = f(g(x))\,g'(x)$

Step 2

$$\int_a^b f(g(x))\,g'(x) = \int_a^b p'(x) \overset{\text{FTOC (int a deriv)}}{=} p(b) - p(a)$$
$$= F(g(b)) - F(g(a))$$
$$= \int_c^{g(b)} f - \int_c^{g(a)} f = \int_{g(a)}^{g(b)} f(u)\,du \qquad \square$$

One more example of separable eq, showing complexities of abs val

Ross ITODE ch 3.2 p.79

Falling body $m\frac{dv}{dt} = F_1 + F_2$

$$\begin{cases} \frac{1}{4}\frac{dv}{dt} = 8 - 2v \\ v(0) = 0 \end{cases} \Rightarrow \frac{1}{8-2v}\,dv = 4dt$$

let $u := 8-2v$, $du = 2dv$

$$-\frac{1}{2}\int \frac{1}{u}\,du = \int 4dt + c$$

$$-\frac{1}{2}\ln|8-2v| = 4t + c$$

$$\ln|8-2v| = -8t + c_1$$

$$|8-2v| = c_2 e^{-8t}$$

plug in $t=0 \Rightarrow v=0$: $8 = c_2 1 \Rightarrow c_2 = 8$

RHS always pos $\Rightarrow$ drop abs val

$$8-2v = 8e^{-8t}$$

$$v = 4(1-e^{-8t})$$

I wondered a lot about abs val signs causing problems e.g.

$$\int \frac{1}{\sin y}\cos y\,dy = \int \frac{x}{x^2+1}\,dx$$

$$\Rightarrow \ln|\sin y| = -\frac{1}{2}\ln|x^2+1|$$

$\sin(y)$ certainly can be neg, but there is nothing to worry about. See my write up of Suskowski CWAG ch7

$\int \frac{1}{u}u'\,du = \ln|u|$ <u>is</u> correct, even if $\frac{u'}{u}$ is neg!

Exact Diff Eqs

See my inserted sheets 6a 6b

Fixed up here

Here we shift gears and follow the better discussion in T & Pollard p.73

Then we introduce vector calculus ideas to make things clearer. such as looking for a Potential fcn.

Given $P(x,y) + Q(x,y)y' = 0$ [or the differential form $P\,dx + Q\,dy = 0$ not really]

<u>on a domain D</u> we hope to find a fcn $\varphi : \underset{\mathbb{R}^2}{D} \to \mathbb{R}$ such that

$$\frac{d}{dx}\varphi(x,y) = \underbrace{D_1\varphi}_{P} + \underbrace{D_2\varphi}_{Q}\,y'$$

The topology of D plays a role in determining if this is possible.

Thm $D = [a,b]\times[c,d]$ rectangle (a more advanced pf would only need it to be simply conn)

P, Q, P_y, Q_x Cont on D

φ is C^2 smooth

$$\left.\begin{array}{l} \exists\ \varphi : D \to \mathbb{R} \ni \\ \varphi_x = P \\ \varphi_y = Q \end{array}\right\} \Longleftrightarrow P_y = Q_x$$

<u>pf</u> ($\Rightarrow$) If φ exists such that $D_x\varphi = P$, $D_y\varphi = Q$ then $D_yP = D_yD_x\varphi$, $D_xQ = D_xD_y\varphi$

and if φ is C^2 then $\varphi_{yx} = \varphi_{xy}$ by equality of mixed partials

Marsden & Tromba VC p.140

NOTE:

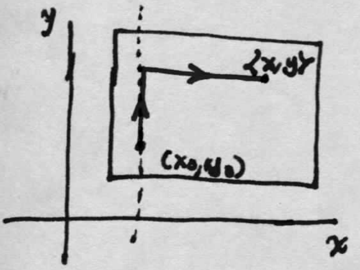
$\frac{d}{dx}\varphi(x,y) = \varphi_x dx + \varphi_y dy = 0$ $\quad \varphi(x, y(x)) = c\ \forall x \in I$

means $\varphi(x,y) = c$ is a <u>conserved quantity</u> or <u>Const of motion</u>

c is the param for a 1 param family of solutions to ODE $y_c(x) \leftarrow c$

cont'd →

($\Leftarrow$) we want $\varphi : D \to \mathbb{R} \ni \quad \varphi_x = P \quad \varphi_y = Q(x,y)$



we define φ by $\frac{\partial}{\partial x}\varphi(x,y) = P(x,y)$ for any $(x,y) \in D$

① Fix some base pt $\langle x_0, y_0\rangle \in D$ and line integrate to get φ horiz red line in picture

$$\varphi(x,y) = \int_{x_0}^{x} P(t,y)\,dt + \Upsilon(y) \leftarrow \text{some fcn which depends only on } y$$

② Then for this φ we must have $\frac{\partial}{\partial y}\varphi = Q$

we need D to be a rectangle in this pf because we need these vertical and horizontal paths for <u>any</u> $\langle x,y\rangle \in D$

$$\frac{\partial}{\partial y}\varphi(x,y) = \frac{\partial}{\partial y}\int_{x_0}^{x} P(t,y)\,dt + \Upsilon'(y) \overset{!}{=} Q(x,y)$$

$$= \int_{x_0}^{x} \underbrace{P_y(t,y)}_{Q_x(t,y)}\,dt + \Upsilon'(y)$$

$$\overset{\text{FTOC}}{=} \cancel{Q(x,y)} - Q(x_0,y) + \Upsilon'(y) = \cancel{Q(x,y)}$$

$$\Upsilon'(y) \overset{!}{=} Q(x_0,y)$$

$$\Rightarrow \Upsilon(y) = \int_{y_0}^{y} Q(x_0,t)\,dt \qquad \text{vertical red line in picture}$$

$$\Rightarrow \boxed{\varphi(x,y) = \int_{x_0}^{x} P(t,y)\,dt + \int_{y_0}^{y} Q(x_0,s)\,ds}$$

if we started with $\frac{\partial}{\partial y}\varphi \overset{!}{=} Q$ we'd get an alogous formula $\int_{y_0}^{y} Q(x,t)\,dt + \int_{x_0}^{x} P(s,y_0)\,ds$

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example Boyce & di Prima p. 36

$$\underbrace{(y\cos x + 2xe^y)}_{P} + \underbrace{(\sin x + x^2e^y + 2)}_{Q}\frac{dy}{dx} = 0$$

D here is $\mathbb{R}^2$
usually we like to take base pt $= \langle 0,0\rangle$ if possible (terms disappear) but call base pt $\langle a,b\rangle$ here

Calculate: $P_y = \cos x + 2xe^y$
$Q_x = \cos x + 2xe^y$ — same, so ODE exact

From the above formula: $\varphi(x,y) = \int_a^x (y\cos x + 2xe^y)\,dx + \int_b^y (\sin a + a^2e^y + 2)\,dy$

$$= y\sin x + x^2e^y \Big]_a^x + y(\sin a) + a^2e^y + 2y \Big]_b^y$$

$$= y\sin x + x^2e^y - \cancel{y\sin a} - \cancel{a^2e^y} + \cancel{y\sin a} + \cancel{a^2e^y} + 2y - [b\sin a + a^2e^b + 2b]$$

$\varphi(x,y) = y\sin x + x^2e^y + 2y + C(a,b) \leftarrow$ const, depends on base pt. wlog we can omit it since the ODE comes from φ_x and φ_y and we have a level surface const anyway!

$\varphi(x,y) = c$ is the implicit soln to ODE — although usually we can't explicitly solve for $y(x)$

1 param family of solutions

$$y\sin x + x^2e^y + 2y = c$$

Lets say we had initial condition $y(0) = 1$. Then $1\sin 0 + 0^2e^1 + 2 = c \Rightarrow c = 2$

This member of the family of solns has param $c = 2$: $y\sin x + x^2e^y + 2y = 2$

□

10/4/2025

6a

What is an Exact ODE?

▷ Given $y = y(x)$ we can define the curve $\sigma(x) = \begin{bmatrix} x \\ y(x) \end{bmatrix}$

Let there be a fcn g that is const on this curve : $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad g(\sigma(x)) = c$

i.e. $g(x, y(x)) = c$

Then $\dfrac{d}{dx} g(\sigma(x)) = \dfrac{d}{dx} c = 0$

$$D_1 g_{\sigma(x)} \sigma_1'(x) + D_2 g_{\sigma(x)} \sigma_2'(x) = 0 \quad \text{or} \quad D_1 g_{xy} \cdot 1 + D_2 g_{xy} y'(x) = 0$$

$$\text{or} \quad \boxed{P(x,y) + Q(x,y)\, y' = 0}$$

$$P(x,y) + Q(x,y) \frac{dy}{dx} = 0$$

$$\text{“} P(x,y)\,dx + Q(x,y)\,dy \text{”} = 0$$

formally, but logically this doesn't make sense, see below

▷ What if we were given

$P(x,y) + Q(x,y)\, y' = 0$ ODE

Because the ODE is then exactly Dg_x, no pieces missing.

we can try to make it exact with integrating factor μ

Does a g fcn exist? If so, this ODE is called **exact** and g is called a conserved quantity, or constant of motion, or 1st Integral

Can we find g? Yes, if $P_y = Q_x$ and the domain is simply conn – more on this later

Can we solve for $y = y(x)$ explicitly? In general, No. We consider finding g as "solving" the ODE.

▷ More generally, we would not have $y = y(x)$ but $\sigma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$$g(\sigma(t)) = c$$

$$\frac{d}{dt} g = D_{\bullet} g_{\sigma(t)}(\dot\sigma(t)) = D_1 g_{\sigma(t)} \dot\sigma_1(t) + D_2 g_{\sigma(t)} \dot\sigma_2(t) = 0$$

$$= P(x,y)\,\dot x + Q(x,y)\,\dot y = 0$$

$$= P(x,y) \frac{dx}{dt} + Q(x,y) \frac{dy}{dt} = 0$$

"mult by dt and get $P\,dx + Q\,dy = 0$"

More rigorously, we can write this in terms of differential forms:

$$P(x,y)\, d\pi_1(\dot\sigma) + Q(x,y)\, d\pi_2(\dot\sigma) = 0$$

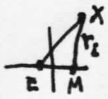
$$P(x,y)\, dx(\dot\sigma) + Q(x,y)\, dy(\dot\sigma) = 0$$

$$(P\, dx(\cdot) + Q\, dy(\cdot))(\dot\sigma) = 0$$

so in this sense we can write $P(x,y)\,dx + Q(x,y)\,dy = 0$ as a diff form

How is g as 1st Integral? A classic example would be the planar Restricted 3 Body Problem

Roy OM p.128 my thesis p.7-8



$$\Omega(x,y) := \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

$$r_i = \left[(x-x_i)^2 + y^2\right]^{1/2} \quad \text{where } r_i = r_E \text{ or } r_M$$

The eqs of motion are

$$\ddot x - 2\dot y - \Omega_x = 0$$

$$\ddot y + 2\dot x - \Omega_y = 0$$

mult 1st by $\dot x$, 2nd by $\dot y$ and add

$$\dot x(\ddot x - 2\dot y - \Omega_x) + \dot y(\ddot y + 2\dot x - \Omega_y) = 0$$

$$\dot x \ddot x + \dot y \ddot y - \Omega_x \dot x - \Omega_y \dot y = 0$$

$$\frac{d}{dt}\left(\dot x^2 + \dot y^2 - 2\Omega\right) = 0$$

$$\boxed{\dot x^2 + \dot y^2 - 2\Omega = c}$$

This is the 1st Integral it is a conserved quantity on the trajectories $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

If we make a COV, it is the Hamiltonian $= g(x(t), y(t), \dot x, \dot y)$ traj in phase space.

10/4/2025

6b

▷ How do Lie derivs get involved?

If I have a smooth ODE $\dot{x} = X(x)$ then for a given pt x_0, in a nbhd $\mathcal{N}_{x_0}$ there is a soln curve $\dot{\varphi}_{x_0}(t) = X(\varphi_{x_0}(t))$

Then if there is a fcn $g : \mathcal{N}_{x_0} \to \mathbb{R}$ we can compute its Lie deriv:

$$(\mathcal{L}_X g)_{x=\varphi_{x_0}(t)} = (g \circ \varphi_{x_0})'(t) = Dg_{\varphi_{x_0}(t)}(\dot{\varphi}_{x_0}(t)) = Dg_x(X(x)) \qquad x = \varphi_{x_0}(t)$$

Now if $g(\varphi_{x_0}(t)) = c$

$$(g \circ \varphi_{x_0})'(t) = 0 \Rightarrow Dg_x(X(x)) = 0$$

And we can also show $Dg_x(X(x)) = 0 \Rightarrow g$ is a conserved quantity

pf. Let $\Psi(t) = g(\varphi_x(t))$

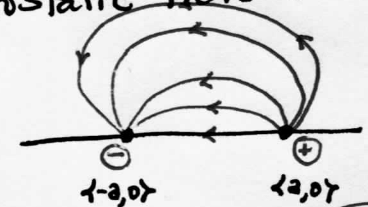
then $\Psi'(t) = Dg_{\varphi_x(t)}(\dot{\varphi}_x(t)) = Dg_{\varphi_x(t)}(X(x)) = 0$ by hypoth

$\Rightarrow \Psi'(t) = 0$ $\qquad Dg_{\varphi_x(t)}(X(\varphi_x(t))) = Dg_{\sigma(t)}(X(\sigma(t))) = 0$

$\Rightarrow \Psi(t) = c \Rightarrow g(\varphi_x(t)) = c$

▷ Now let's consider a specific example — Electrostatic field of 2 pt charges equal and opposite

$$\dot{x} = \vec{E}(x) \qquad E(x) = \frac{kq}{\|x-a\|^3}(\vec{x}-\vec{a}) + \frac{k\cdot(-q)(\vec{x}-\text{-}\vec{a})}{\|x-\text{-}a\|^3}$$



bot is mirror image

$x = (x,y)$

$$\text{So } \frac{1}{kq}E^{(1)} = \frac{(x-a)}{N^{3/2}} - \frac{(x+a)}{P^{3/2}} \qquad \text{where } N := [(x-a)^2 + y^2]$$

$$\frac{1}{kq}E^{(2)} = \frac{y}{N^{3/2}} - \frac{y}{P^{3/2}} \qquad P := [(x+a)^2 + y^2]$$

This is from Smythe Static Dynamic Elec P.7-8 Stored in Feynman ch 4 sheets

<u>Claim</u>: $C(x,y) := \dfrac{(x+a)}{P^{1/2}} - \dfrac{(x-a)}{N^{1/2}}$ is a 1^{st} Integral or Const of Motion

This means $DC_x\left(\begin{bmatrix} E^1 \\ E^2 \end{bmatrix}\right) = 0 \qquad DC_x = [C_x \;\; C_y]$

$$C_x = \frac{y^2}{P^{3/2}} - \frac{y^2}{N^{3/2}} \qquad C_y = \frac{-(x+a)y}{P^{3/2}} + \frac{(x-a)y}{N^{3/2}}$$

$$\left[\frac{y^2}{P^{3/2}} - \frac{y^2}{N^{3/2}}, \; \frac{-(x+a)y}{P^{3/2}} + \frac{(x-a)y}{N^{3/2}}\right]\begin{bmatrix} \frac{(x-a)}{N^{3/2}} - \frac{(x+a)}{P^{3/2}} \\ \frac{y}{N^{3/2}} - \frac{y}{P^{3/2}} \end{bmatrix} =$$

$$y^2\left(\frac{N^{3/2} - P^{3/2}}{P^{3/2}N^{3/2}}\right)\left(\frac{P^{3/2}(x-a) - N^{3/2}(x+a)}{P^{3/2}N^{3/2}}\right) + \left(\frac{P^{3/2}(x-a)y - N^{3/2}(x+a)y}{P^{3/2}N^{3/2}}\right)\left(\frac{y(P^{3/2} - N^{3/2})}{P^{3/2}N^{3/2}}\right)$$

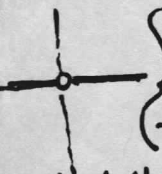
$$= y^2\left[\frac{(N^{3/2} - P^{3/2})(P^{3/2}(x-a) - N^{3/2}(x+a)) + (P^{3/2} - N^{3/2})(P^{3/2}(x-a) - N^{3/2}(x+a))}{(P^{3/2}N^{3/2})^2}\right] = y^2\left[\frac{AB - AB}{P^3N^3}\right] = 0 \checkmark$$

So C is indeed a 1^{st} Integral □

Tenenbaum & Pollard ODE p. 80-81 has a good discussion about working backward from a potential fcn φ to find the ODE form that it generates

$\varphi(x,y) = c$		ODE $\varphi_x dx + \varphi_y dy$
$xy = c$	$\Rightarrow$	$y\,dx + x\,dy = 0$
$x^2 y = c$	$\Rightarrow$	$2xy\,dx + x^2\,dy = 0$
$y \sin x = c$		$y \cos x\,dx + \sin x\,dy = 0$
$\ln(xy) = c$		$\frac{1}{x}dx + \frac{1}{y}dy = 0$
$\frac{y}{x} = c$		$-\frac{y}{x^2}dx + \frac{x}{x^2}dy = 0$
$\mathrm{Tan}^{-1}\left(\frac{y}{x}\right) = c$	$\Rightarrow$	$\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 0$

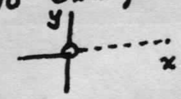
Many more examples in book.



Both of these are only defined on $\mathbb{R}^2 - \{y \text{ axis}\}$

~~may not always~~ not guaranteed to exist

This is the famous example showing that φ ~~cannot~~ exist unless the domain is simply connected. ODE defined on $\mathbb{R}^2 - \{0\}$ but for φ to exist, we need a branch cut like

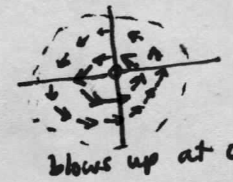


Let's work this last example in detail [From Munkres AOM p.260-261]

Define $\omega = \left(\frac{-y}{x^2+y^2}\right)dx + \left(\frac{x}{x^2+y^2}\right)dy$, with $P = \frac{-y}{x^2+y^2}$, $Q = \frac{x}{x^2+y^2}$

This is exact on $\mathbb{R}^2 -$ (half x axis) but NOT on $\mathbb{R}^2 - \{0\}$

As a vf $\vec{F} = \begin{bmatrix} P \\ Q \\ 0 \end{bmatrix} \leftarrow$ zero it is



blows up at origin

ⓐ show $P_y = Q_x$ (This is $\nabla \times F = 0$ for $F = \begin{bmatrix} P \\ Q \\ 0 \end{bmatrix}$)

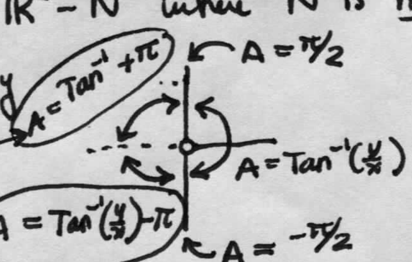
$$\frac{\ell h' - h\ell'}{\ell^2} \qquad P_y = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{\cdot} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

SAME

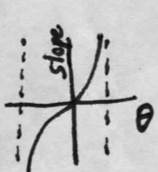
$$Q_x = \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

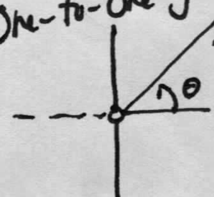
ⓑ we can modify (extend) ~~Tan~~ $(x,y) \longmapsto$ ~~R~~ $\mathrm{Tan}^{-1}(y/x)$ so that this fcn is defined on $\mathbb{R}^2 - N$ where N is neg ~~x~~ x axis and O. A is One-to-One globally.

Define $A(x,y)$ by



maybe should have called this θ





$m = \frac{y}{x}$ $\tan\theta = \text{slope} = m$

we map $(x,y) \mapsto \frac{y}{x} = m$ and then $m \mapsto \overset{\text{angle}}{\theta}$ with ~~[illegible]~~ the principal branch of the angle fcn specified by $\theta = 0$ on pos x axis

This fcn A is continuous and ~~especially~~ in fact C^∞

How do we know A is smooth? Munkres gives an inv fcn thm arg from the polar co-ord map

$$\mathrm{Tan}^{-1}\left(\frac{y}{x}\right) := \int_0^{y/x} \frac{1}{1+t^2}\,dt$$

$$dT = D_x T\,dx + D_y T\,dy$$

Leibniz rule

$$D_x T = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} \cdot \frac{1}{\cancel{x^2}}$$

$$D_y T = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} \overset{\cancel{x^2}}{=} \frac{x}{x^2+y^2}$$

and we recover ω

con'td

Thus we showed $dA = \omega$ on $\mathbb{R}^2 - N$. Thus, since $\omega = 0$
$dA = 0 \Rightarrow A(x,y) = c$
and this is the 1 param family of solutions to ODE on $\mathbb{R}^2 - N$.

(c) Can there be some other fcn f that satisfies ODE (PDE?) and be defined on $\mathbb{R}^2 - \{0\}$? NO.

Suppose f did exist.

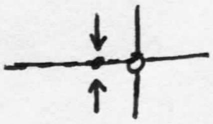
<u>On $\mathbb{R}^2 - N$</u> (open set) we have $df = \omega = dA$

$\Rightarrow df - dA = 0 \Rightarrow d(f-A) = 0$

$\Rightarrow f(x,y) - A(x,y) = c$

$\Rightarrow f(x,y) = A(x,y) + c$ <u>on $\mathbb{R}^2 - N$</u>

What is f on N? Since it must be dif'b, it must be continuous there.
Pick a pt, say $(-1,0)$



we must have $\lim\limits_{y \uparrow 0} f(-1,y) \overset{!}{=} \lim\limits_{y \downarrow 0} f(-1,y)$

$$\Rightarrow \underbrace{\lim_{y\uparrow 0} \theta(-1,y)}_{-\pi} + \cancel{c} \overset{!}{=} \underbrace{\lim_{y \downarrow 0} \theta(-1,y)}_{+\pi} + \cancel{c}$$

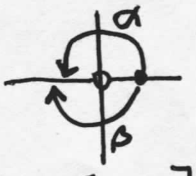
f is not continuous $\Rightarrow\Leftarrow$

The heart of the matter is that $\mathbb{R}^2 - \{0\}$ is not simple conn.
Thus we can have 2 separate paths α, β from the same base pt to the same target pt, but $\int_\alpha \omega \neq \int_\beta \omega$ $\quad \int_\alpha F \cdot ds \neq \int_\beta F \cdot ds$

because of the singularity the loop $\alpha + -\beta$ is not contractible

$\omega = F \cdot ds$ (If they were, we would get the same answer for line integral)

However lets take $\omega = \frac{-y}{x^2}dx + x\,dy = 0$ this is also not defined at 0 and has $\varphi(x,y) = \frac{y}{x}$



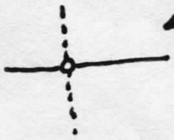
But we don't run into the same difficulty (that is why the angle fcn (vf) is special)

$$\int_\alpha \omega = \int_\alpha \vec{F} \cdot d\vec{s} = \int_0^\pi \frac{r \sin t}{r^2 \cos^2 t}$$

$\alpha(t) = \begin{bmatrix} r\cos t \\ r \sin t \end{bmatrix}$ $0 \le t \le \pi$

$\alpha'(t) = \begin{matrix} -r\sin t \\ r \cos t \end{matrix}$

I think this isn't worth doing. The domain is



NOT even <u>Conn</u>!

p.40

Integrating Factors

Now we want to investigate $M\,dx + N\,dy$ not being exact, but it can be made exact by multiplying by a fcn $\mu(x,y)$:

$$\mu(M\,dx + N\,dy) = 0$$

Then we would have $\frac{\partial}{\partial y}(\mu M) \overset{!}{=} \frac{\partial}{\partial x}(\mu N)$

i.e. $\mu_y M + \mu M_y = \mu_x N + \mu N_x$

$$\Rightarrow \mu_y M - \mu_x N + (N_y - N_x)\mu = 0$$

Ross ITODE Claims integrating factors always exist, but in general, finding them is as hard as solving the ODE.

You have to solve this PDE

by the way, this eq is homog.

Let's look at some examples

(1) $\underbrace{(y^2 + xy)}_{M}dx \underbrace{+ -x^2}_{N} dy = 0$

First we see $M_y = 2y + x \neq -2x = N_x$ so not exact.

Take $\mu = \frac{1}{xy^2} \Rightarrow \left(\frac{y^2+xy}{xy^2}\right)dx + \left(\frac{-x^2}{xy^2}\right)dy = 0$ provided $x \neq 0$ and $y \neq 0$

$$= \left(\frac{1}{x} + \frac{1}{y}\right)dx + \left(-\frac{x}{y^2}\right)dy = 0$$

and now $M_y = -y^{-2}$ and $N_x = -y^{-2}$ so ODE is exact

$$\varphi(x,y) = \int^x M\,dx + \int^y N\,dy - \iint^{y\,x} M_y\,dx\,dy$$

$$\int M\,dx = \int^x \left(\frac{1}{x} + \frac{1}{y}\right)dx = \ln|x| + \frac{x}{y}$$

$$\int N\,dy = \int \frac{-x}{y^2}dy = +\frac{x}{y} \quad \text{and} \quad \iint y^{-2}\,dx\,dy = \frac{x}{y}$$

$$\Rightarrow \varphi(x,y) = \ln|x| + \frac{x}{y} + \cancel{\frac{x}{y}} - \cancel{\left(\frac{x}{y}\right)}$$ and we must exclude $x=0$, $y=0$.

(2) Most useful situation: μ is a fcn of x only or y only.

Lets say μ depends only on x:

$$\Rightarrow \frac{\partial}{\partial y}(\mu M) = \mu M_y \quad \text{and} \quad \frac{\partial}{\partial x}(\mu N) = \mu N_x + N\frac{d\mu}{dx}$$

$$\Rightarrow \mu M_y \overset{!}{=} \mu N_x + \frac{d\mu}{dx}$$

$$\Rightarrow \frac{d\mu}{dx} = \mu\left(\frac{M_y - N_x}{N}\right)$$

if $\frac{(M_y - N_x)}{N}$ is a fcn of x only, $\exists\ \mu = \mu(x)$ which solves this.

To see this, let's consider the linear ODE which Boyce & diPrima introduced first:

Linear ODE

~ here integrating factor is nice.

Consider the special form $\boxed{\frac{dy}{dx} + p(x)\,y = g(x)}$ or $L(y) = g$ where linear operator $L = D_x + pI$

[we could express this in our previous std forms by $y' = \underbrace{g(x) - p(x)\,y}_{f(x,y)}$ or $(p(x)\,y - g(x))\,dx + 1\,dy = 0$]

How could we solve it?

Product rule trick $\frac{d}{dx}(Py) = P'y + Py'$ where $P' = p$ but this can't work because this also makes $P = 1$

Try multiplying both sides by integrating factor μ (to be determined):

$$y' + py = g$$

$$\underbrace{\mu y' + \mu p y}_{} = \mu g$$

Now we want $= \frac{d}{dx}(\mu y)$ In order for that to happen: $\mu' y + \cancel{\mu y'} \overset{!}{=} \cancel{\mu y'} + \mu p y$

$$\Rightarrow \mu' \cancel{y} \overset{!}{=} \mu p \cancel{y}$$

$$\Rightarrow \frac{d\mu}{dx} = \mu p \quad \text{separable eq}$$

$$\Rightarrow \int \frac{1}{\mu}\,d\mu = \int p\,dx + c \quad \text{but we don't need } c$$

$$\ln|\mu| = \int p\,dx$$

$$|\mu| = e^{\int p\,dx} \quad \text{drop abs val since } e^x > 0\ \forall x$$

$$\boxed{\mu(x) = e^{\int p\,dx}}$$

Ross has a nice derivation:
Want $(py - g)dx + 1dy$ to be exact
Multiply by μ which depends only on x
$\mu(py-g)dx + \mu dy = 0$
Must have $M_y = N_x$:
$\overset{M_y}{\mu p} \overset{!}{=} \overset{N_x}{\mu_x} \Rightarrow \frac{d\mu}{dx} = \mu p$
μ fcn of x only

Now let's solve the ODE:

$$\mu y' + \mu p y = \mu g$$

$$\frac{d}{dx}(\mu y) = \mu g$$

$$\int d(\mu y) = \int \mu(x) g(x)\,dx + c$$

$$\mu y =$$

$$y = \mu^{-1} \int \mu(x) g(x)\,dx + \mu^{-1} c = e^{-\int p\,dx} \int^x e^{\int^s p\,dx} g(s)\,ds + c e^{-\int p\,dx}$$

Let's do an example: $y' - 2xy = x$, $y(0) = 1$ $\quad \mu(x) = e^{\int(-2x)dx} = e^{-x^2}$

p.17

$$e^{-x^2}(y' - 2xy) = e^{-x^2} x$$

$$\frac{d}{dx}\left(y e^{-x^2}\right) = x e^{-x^2}$$

$$y e^{-x^2} = \int e^{-x^2} x\,dx \overset{\substack{u = -x^2 \\ du = -2x\,dx}}{=\!=\!=} -\frac{1}{2} e^{-x^2} + c$$

$$y(x) = e^{x^2}\left[-\frac{1}{2} e^{-x^2} + c\right] = c e^{x^2} - \frac{1}{2}$$

plug in IC $1 \overset{!}{=} c + -\frac{1}{2}$, $c = \frac{3}{2}$

$$y(x) = \frac{3}{2} e^{x^2} - \frac{1}{2} \qquad \square$$

we also need to discuss Bernoulli eq later