Ch 2.2

This is the big deal that gives us ALL solns

Thm 2.1 $\left.\begin{array}{l} p, g \text{ continuous on } [\alpha,\beta] \\ x_0 \in [\alpha,\beta] \\ y_0 \text{ arb} \in \mathbb{R} \end{array}\right\}$. $\Rightarrow \exists!$ fcn $y : [\alpha,\beta] \to \mathbb{R} \ni$

$\boxed{y' + py = g}$ on $[\alpha,\beta]$

and $y(x_0) = y_0$

pf for more general ODE given in §2.13

Pab # 2.13

p.24

Bernoulli Eq $y' + py = gy^n$.

ODE that we can reduce to Integrating Factor form.

if $n=0$ $y' + py = g$ and we use ch 2.1 integrating factor

if $n=1$ $y' + py = gy \Rightarrow y' + (p-g)y = 0$ again we could use integrating factor

OR $\frac{dy}{dx} = -(p-g)y$

$\frac{1}{y}dy = (g-p)\,dx$

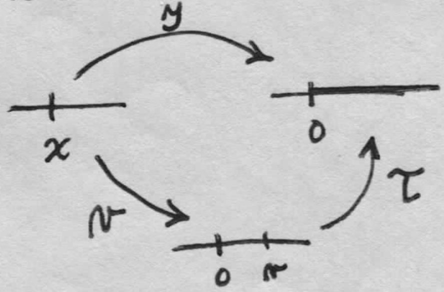
Now to the real issue $n \geq 2$

The pattern of these substitution problems is to "factor"

The pattern of these substitution problems is to "factor" a known transform

$y(x) = \tau \circ v(x)$ when τ is a known transform

here $\tau(v) = v^{\frac{1}{-n+1}}$ so $v \geq 0$



Then the ODE $y' = f(y)$

becomes $v'(x) = \frac{1}{\tau'(v)} f(\tau(v))$

which *hopefully* is easier to solve. Then recover y by $y = \tau \circ v$

$v(x) = \frac{1}{y(x)^{n-1}}$

But we can't have $y(x)=0$ and since y is cont, we always have $y>0$ or $y<0$ on an interval

(iv) example $x^2y' + 2xy = 1y^3$

$\tau(v) = v^{\frac{1}{-3+1}} = v^{\frac{1}{-2}} = \frac{1}{\sqrt{v}}$

$y = v^{-1/2}$

$y' = -\frac{1}{2}v^{-3/2}v'$

plug this in

$x^2\left(-\frac{1}{2}v^{-3/2}v'\right) + 2x\,v^{-1/2} = v^{-3/2}$

mult by $v^{3/2}$ $-\frac{1}{2}x^2v' + 2xv = 1$ and assuming $x \neq 0$

$\frac{-2}{x^2}$ $\boxed{v' + -\frac{4}{x}v = \frac{-2}{x^2}}$ Now we should be able to solve this.

$$\overset{p}{v' - \frac{4}{x} v} = \frac{-2}{x^2} \qquad \mu = e^{\int p} =$$

$$\int p\,dx = -4\int^x \frac{1}{x}\,dx = -4\ln x = \ln x^{-4}$$

$$\Rightarrow \frac{d}{dx}\left(v c x^{-4} \right) = c x^{-4}\left(\frac{-2}{x^2}\right) \qquad c e^{-4\ln x} = c e^{\ln x^{-4}} = \frac{c}{x^{+4}}$$

$$= \frac{-2c}{x^6}$$

$$x^n \rightsquigarrow \frac{1}{n} x^{n-1}$$

$$n x^n \leftsquigarrow x^{n-1}$$

$$v \cancel{c} x^{-4} = -2\int x^{-6}\,dx = = \left(-5x^{-5} + c\right)$$

$$-2$$

$$x^4 \; v x^{-4} = \left(10x^{-5} + c\right) x^4$$

$$v = \frac{10}{x} + c x^4 \frac{x}{x} \qquad y = v^{-1/2} \Rightarrow y^2 = v^{-1}$$

$$y^{-2} = v$$

$$\frac{1}{y^2} = \frac{10 + c x^5}{x}$$

$$y^2 = \frac{x}{c x^5 + 10}$$

$$y = \pm\sqrt{\frac{x}{c x^5 + 10}}$$

p. 438

Book says $y = \left(\frac{2}{5} x^{-1} + c x^2\right)^{-1/2}$

Ch 2.3 Non-Linear Eqs

Cauchy IVP

<u>Thm 2.2</u> R rectangle $[\alpha;\beta]\times[\gamma,\delta]$
f, $D_2 f$ Continuous on R
$(x_0, y_0) \in R$

$$\Rightarrow \begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

$\exists!\; y$ dif'b in some interval $[x_0 - h,\; x_0 + h]$

(Linearity gave us whole interval)

ex 1 <u>Bad behaviour of n linear probs</u>

$y' = y^2$
$y(0) = 1$

By direct subs, we see $y(x) = \frac{1}{1-x}$ is a soln.

singularity at $x=1$, so valid on $[-\infty, 1]$ but nothing special about $x=1$ in ODE: $y' = y^2$

But if we replace IC by $y(0) = 2$
Then we can see $y(x) = \frac{2}{1-2x}$ is a soln with singularity at $x = 1/2$

$\Rightarrow$ SINGULARITIES can depend on initial conds!

ex 2 Relax conds in thm 2.2 and get non-uniqueness!

~~12~~

$C^n(I \subseteq \mathbb{R})$ is a VS over $\mathbb{R}$

12/19/2014

[redacted]

①

Ch 2.1 Linear Eqs (integrating factor)
2.2 Further Discussion of Linear Eqs
Existence & Uniqueness
2.3 Nonlinear Eqs
2.4 Separable Eqs
2.5 Exact Eqs
2.6 Integrating factors

2.7 Homog Eqs
2.8 Misc Probs
2.9 Growth, Decay, Mixing probs
2.10 O.G. Trajecto and other geometrical probs.
2.11 Elementary Mechanics
2.12 Existence and Uniq Thm

kusi

$$L(y) = \frac{d}{dx}(\cdot) + pI(\cdot)$$
$$= y' + py$$

$y \in$ VS fcns over $\mathbb{R}$

provided α is a const wrt x

$$L(\overbrace{\alpha f}+g) =$$
$$\alpha L(f) = \alpha(f' + pf)$$
$$L(g) = g' + pg$$
$$\parallel$$
$$(\alpha f + g)' + p(\alpha f + g)$$
$$\alpha f' + g' + \alpha p f + pg$$
$$\alpha L(f) + L(g)$$

Variation of Param

#13 p. 19 $\quad y' + p(x)y = g$

ⓐ if $g = 0$ show soln
$$y(x) = \underset{\uparrow \text{const}}{A}\, e^{-\int p(x)dx}$$

ⓑ If $g \neq 0$
$$y(x) = \underset{\text{fcn}\,\downarrow}{A(x)}\, e^{-\int p(t)dt} = u(x)\, y_1(x)$$
← soln to homog problem

Show A must satisfy $A'(x) = g(x)\, e^{\int p(x)dx}$

$$y = u y_1$$
$$y' = u' y_1 + u y_1'$$

[redacted]

plug into eq:
$$y' + py = g$$
$$u'y_1 + uy_1' + p(uy_1) \overset{!}{=} g$$
$$u'y_1 + u\underbrace{(y_1' + py_1)}_{=0} = g$$
$$\Rightarrow u'y_1 = g$$
$$\Rightarrow u' = \frac{g}{y_1} = \frac{g}{ae^{-\int p}} = g(x)\, e^{\int p\, dx} \quad \square$$

General form: $F(x, y, y', y'') = 0$ too hard

we only consider $y'' = f(x, y, y')$ ⊛

$\boxed{\text{Thm 3.1}}$ If $D_1 f, D_2 f, D_3 f$ are continuous in region R pt $(x_0, y_0, y_0') \in R$ $\Big\} \Rightarrow$ ⊛ has a unique solution in nbhd $y = y(x)$ of pt (x_0, y_0, y_0') satisfying ICs $y(x_0) = y_0$, $y'(x_0) = y_0'$

General forms for LINEAR ODE:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

[we assume P, Q, R, G cont in interval and P nonzero in this interval]

Simple example: mass on a spring $mu'' + cu' + ku = F(t)$

Legendre: $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

Bessel: $x^2 y'' + xy' + (x^2 - n^2)y = 0$

We can attempt non-linear problems with special form $y'' = f(x, y')$ ← no y

$y'' = f(y, y')$ ← no x

we shall tackle

$$y'' + p(x)y' + q(x)y = g(x) \quad ⊛⊛$$

that is to say

$$f(x, y, y') = -p(x)y' - q(x)y + g(x)$$

cf Avez DG p.82

$\boxed{\text{Thm 3.2}}$ p, q, g cont on interval (α, β)

$\Rightarrow \exists! \overset{\text{fcn}}{y} \ni y$ satisfies ⊛⊛ on (α, β) for ICs $y(x_0) = y_0$, $y'(x_0) = y_0'$

ⓔⓧ $\left.\begin{array}{l} y'' + y = 0 \\ y(0) = 0 \\ y'(0) = 0 \end{array}\right\}$ This has solns ~~$y(x) = \cos x$~~ $y(x) = \sin x$ ICs rule this out

Here we are not attempting BVPs $y(a) = A$, $y(b) = B$

P.86 (see cheney) 12/22/2014

①

Ch 3.2 Fund Soln's of Homog Eq

Diff operatr $L_{pq}: C^2[\alpha,\beta] \longrightarrow C[\alpha,\beta]$

$$y \longmapsto y'' + py' + qy = 0$$

Thm 3.3 If y_1 and y_2 are each soln's of $L(y) = 0$ then so is $cy_1 + y_2$

$\boxed{L \text{ is linear}}$ ← This gives us Superposition principle (can make LCs of solns).

$\{y_1, y_2\}$ are a Fundamental Set of Slns (Basis)

if every soln $y = c_1 y_1 + c_2 y_2$

$\boxed{\text{Here is the idea:}}$ we're claiming y_1, y_2 are Fund Solns if — basis for solution space.

for any $y \in L(y) = 0$ we can write $\boxed{y = c_1 y_1 + c_2 y_2}$ ⊛

But once we specify the initial conds $\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \end{cases}$

there can only be one soln possible (call it y^*)

Then if ⊛ is still to be valid, we must have

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= y^*(x_0) \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= (y^*)'(x_0) \end{aligned} \quad \text{i.e.} \quad \underbrace{\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}}_{\det \neq 0} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

we could call this L_{pq}

Thm 3.4 $p, q \in C[\alpha,\beta]$ $\quad C([\alpha,\beta] \to \mathbb{R})$

y_1 and y_2 are soln's to $L(y) = 0$

$\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \neq 0 \;\; \forall\, x \in (\alpha,\beta)$

$\Longrightarrow$ $y \ni$ any soln $L(y) = 0$ can be expressed as LC of y_1, y_2

Wronskian $W(y_1, y_2)$

Thm 3.5 Same conds as above $\Longrightarrow W(y_1, y_2) \equiv 0$ on (α,β)

or $W(y_1, y_2) \neq 0 \;\forall\, x \in (\alpha,\beta)$

$

Thm 3.4 Given y_1, y_2 solns to $L_{pq}(y) = 0$ and $W(y_1, y_2) \neq 0 \;\forall\, x \in (\alpha,\beta)$ $\Longrightarrow$ $\{y_1, y_2\}$ is a basis for all solns.

Thm 3.5 we don't need to worry about whole interval, if $W(x) \neq 0$ for any x, then $W \neq 0 \;\forall x$.

I need to do

$\int \frac{1}{x}\,dx = \ln x$

why $\ln|x|$?

homogeneous

$y = v x$

ch 3.2

p.89

ASIDE

Bot margin of p.88

Following p.254, if we define $y^{①} = y$

$y^{②} = \dot{y}$

Then our 2nd order ODE becomes the system

$$\begin{bmatrix} \dot{y}^{①} \\ \dot{y}^{②} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} y^{①} \\ y^{②} \end{bmatrix}$$

Then if we have 2 soln's Y_1 Y_2

$$Y_i = \begin{bmatrix} y_i^{①} \\ y_i^{②} \end{bmatrix} = \begin{bmatrix} y_i \\ \dot{y}_i \end{bmatrix}$$ and we see we get the same Wronskian on p.264 as here

$$\begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix} = W = \begin{vmatrix} y_1^{①} & y_2^{②} \\ y_1^{②} & y_2^{②} \end{vmatrix}$$

p.88

Word Law

Pf of Thm 3.5

we know y_1 and y_2 satisfy

Row1 $y_1'' + p y_1' + q y_1 = 0$

$L(y_1) = 0$

$L(y_2) = 0$

Row2 $y_2'' + p y_2' + q y_2 = 0$

mult $-y_2 R_1 + y_1 R_2 = (y_1 y_2'' - y_2 y_1'') + p\underbrace{(y_1 y_2' - y_1' y_2)}_{W_{12}(x)} = 0$

Observe $\frac{dW_{12}}{dx} = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'}$

$= y_1 y_2'' - y_1'' y_2$

Thus we have

$$\frac{dW}{dx} + pW = 0$$

Solve this

$\frac{dW}{dx} = -pW$

$\Rightarrow \frac{1}{W}\,dW = -p\,dx$

$\Rightarrow \ln W = -\cancel{px} + c$

$\boxed{W = c e^{-px}}$

NO $\int p(x)\,dx$

p. 90 ③

Abel's Identity $W_{12}(x) = C e^{\int p \, dx}$

Since $e^{(\cdot)} \neq 0$ (unless $x = -\infty$) we see $W \neq 0$

□

$\boxed{\text{Thm 3.7}}$ $p, q \in C((a,b) \to \mathbb{R}) \Rightarrow \exists$ <u>fundamental solns</u> of $L(y) = 0$

<u>pf</u> let $c \in (\alpha, \beta)$

<u>Thm 3.2</u> $\exists!$

$$y_1'' + p y_1' + q y_1 = 0 \qquad y(c) = 1 \qquad y'(c) = 0$$

$$y_2'' + p y_2' + q y_2 = 0 \qquad y(c) = 1 \qquad y'(c) = 0$$

ġ $\tilde{\Omega}$

$$W(y_1, y_2)\Big|_{x=c} = \det \begin{vmatrix} y_1(c) & y_2(c) \\ y_1'(c) & y_2'(c) \end{vmatrix} = \det \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

un proved

Betrayl

Thm 3.1 Says $\exists!$ soln y satisfying ICs on $y(0)$, $y'(0)$
or 3.2

So we are just trying to find a pt x_0 $\ni$ $W(x_0) \neq 0$
Then we know it is nonzero throughout interval.

<u>Thm 3.7</u> p, q cont on $[a,b]$ $\Rightarrow$ $\exists$ basis for soln space $\{y_1, y_2\}$

Then Thm 3.2 gives us the existence of Solns to $L(y) = 0$ for both sets of ICs $\begin{cases} y(c) = 1 \\ y'(c) = 0 \end{cases}$ call this soln y_1

Should call the pt x_0

and $\begin{cases} y(c) = 0 \\ y'(c) = 1 \end{cases}$ call this y_2

then

$$W(y_1, y_2)\Big|_{x=c} = \det \begin{bmatrix} y_1(c) & y_2(c) \\ y_1'(c) & y_2'(c) \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

alternate Pf of Thm 3.6

Does not rely on existence & uniqueness thm

↓

- L is linear
- Solns $\{y_1, y_2\}$ form a basis for soln space if $W(y_1, y_2) \neq 0$ for any pt in (a, b)
- We can always know $\exists$ a soln ~~set~~ basis $\{y_1, y_2\}$.

[redacted]

Suppose y_1, y_2 are solns of $L(y) = 0$ and $W(y_1, y_2) \neq 0$ on (α, β)

To show any other soln $y_3 = c_1 y_1 + c_2 y_2$

Do the p.89 trick

$$\left.\begin{array}{l} L(y_1) = 0 \\ L(y_2) = 0 \\ L(y_3) = 0 \end{array}\right\} \Rightarrow \frac{dW_{12}}{dx} + p W_{12} = 0$$

$$\frac{dW_{23}}{dx} + p W_{23} = 0$$

$$\frac{dW_{13}}{dx} + p W_{13} = 0$$

$$W_{12} = y_1 y_2' - y_1' y_2 = K_{12} e^{-\int p\,dx}$$

$$W_{23} = y_2 y_3' - y_2' y_3 = K_{23} e^{-\int p\,dx} \quad -P$$

$$W_{13} = y_1 y_3' - y_1' y_3 = K_{13} e^{-\int p\,dx}$$

Baby, maybe you're wrong — but, you know, ~~that's~~ it's all right.

and we know $K_{12} \neq 0$ because by hypoth $W_{12} \neq 0$

NOW:

$$y_2 W_{13} - y_1 W_{23} = \underbrace{(y_1 y_2' - y_1' y_2)}_{W_{12}} y_3 = y_2 (K_{13} e^{-P}) - y_1 (K_{23} e^{-P})$$

$$= (K_{12} \cancel{e^{-P}}) y_3 \qquad = (K_{13} y_2 - K_{23} y_1) \cancel{e^{-P}}$$

$$\Rightarrow \boxed{y_3 = \frac{-K_{23}}{K_{12}} y_1 + \frac{K_{13}}{K_{12}} y_2}$$

$\Rightarrow y_3$ is a LC of y_1 and y_2

$\square$

See cheng to understand why we care.

12/22/2014

p.94

⑰ Exact 2^{nd} order Eqs $\quad P(x)y'' + Q(x)y' + R(x)y = 0$

is **EXACT** if it can be written

$$(Py')' + (fy)' = 0$$ where f is some fcn of P, Q, R

~~then~~ i.e $\quad \frac{d}{dx}\left[Py' + fy \right] = 0$

$\Rightarrow \quad Py' + fy = c \quad$ then this is a 1^{st} order linear eq and we solve it by integrating factor.

We want to show the nec cond for exactness is: $\quad P'' - Q' + R = 0$

(it is also sufficient)

⑱ Any 2^{nd} order linear homog eq can be made exact by multiplying by integrating factor $\mu(x)$:

We want $\mu \ni \quad \mu P y'' + \mu Q y' + \mu R y = 0$

$$(\mu P y')' + (fy)' = 0$$

equate coeffs, eliminate f,

μ must satisfy $\quad P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0$

$\boxed{\text{ADJOINT EQ}}$

Find adjoint for:

Bessel: $x^2 y'' + xy' + (x^2 - n^2)y = 0$

Legendre: $(1 - x^2)y'' -$

⑳ Self adjoint $\Rightarrow$ adj eq = orig eq. $\quad$ Nec cond is $P' = Q$

Ch 3.3 Linear Independence
p.95

$\{f, g\}$ is LD if $\exists\ c_1, c_2$ (not both 0) $\ni\ c_1 f + c_2 g = 0$
$\forall x \in (\alpha, \beta)$

Thm 3.8 $\{f, g\}$ LD on $(\alpha,\beta) \Rightarrow W(f,g) \equiv 0$ on (α,β)

Conversely

$$\left.\begin{array}{l} W(f,g) \equiv 0 \text{ on } (\alpha,\beta) \\ \text{One fcn is nowhere } 0 \text{ on } (\alpha,\beta) \end{array}\right\} \Rightarrow \{f,g\} \text{ LD on } (\alpha,\beta)$$

pf $\quad c_1 f + c_2 g = 0$

Suppose w.l.o.g. say $c_1 \neq 0$

then if $c_2 = 0$, $f \equiv 0$ DONE

So assume $c_2 \neq 0$ also

then $f = \frac{-c_2}{c_1} g = Kg$ and $f' = Kg'$

$$W(f,g) = \det\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \det\begin{vmatrix} Kg & g \\ Kg' & g' \end{vmatrix} = K \det\begin{vmatrix} g & g \\ g' & g' \end{vmatrix} = 0$$

Conversely

\$ $g(x) \neq 0 \ \forall x \in I$ → If g has isolated zeros, break I up into subintervals and do this arg on each.

$$W(f,g) = 0$$

$$\det\begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - gf' = 0$$

mult by $\frac{1}{g^2}$ $\quad \dfrac{fg' - gf'}{g^2} = 0$

$$= \frac{d}{dx}\left(f/g\right) \Rightarrow f/g = c$$

$$\Rightarrow f = cg \qquad \text{LD.}$$

we have established

Thm 3.9 p, q Cont on (α,β)
y_1, y_2 LI soln's of $L(y) = 0$ $\Rightarrow$ any soln y is LC of $\{y_1, y_2\}$.

p. 98-99-100

12/22/2014

ch 3.4 Reduction of Order (D'Alembert)

should be called 'compound soln' or something like that.

Suppose we somehow know one soln y_1 to $L(y) = 0$

can we determine a fcn $v \ni vy_1$ is a soln too?

Define $y(x) = v(x)\, y_1(x)$

Then
$$y' = v'y_1 + vy_1'$$
$$y'' = v''y_1 + v'y_1' + v'y_1' + vy_1''$$
$$= vy_1'' + 2v'y_1' + v''y_1$$

Plug this into $L(y)=0$ (we are assuming y is a sol'n)

$$y'' + py' + qy \overset{!}{=} 0$$

$$vy_1'' + 2v'y_1' + v''y_1 + p(v'y_1 + vy_1') + qvy_1 \overset{!}{=} 0$$

$$v\underbrace{(y_1'' + py_1' + qy_1)}_{0} + v'(2y_1' + py_1) + v''y_1 = 0$$

$$\Rightarrow v'' + v'\left(2\frac{y_1'}{y_1} + p\right) = 0$$

let $z := u'$

$$\Rightarrow \frac{dz}{dx} + \left(2\frac{y_1'}{y_1} + p\right)z = 0$$

separable

$$\int \frac{1}{z}\, dz = -\int \left(p + 2\frac{y_1'}{y_1}\right) dx$$

$$\int p + 2\underbrace{\int \frac{1}{y_1} y_1' dx}_{\int \frac{1}{u} du \rightsquigarrow 2\ln u + c_1}$$

$$z = c\,e^{-\int (p + 2\frac{y_1'}{y_1})}$$

Call this $U(x)$

then $c\, e^{-\int p dx} e^{-2\ln u}$ " u^2

$$\Rightarrow z(x) = c\, \frac{1}{y_1^2} e^{-\int p\, dx}$$

$\frac{dv}{dx}$ "

then
$$v(x) = c\int U(t)\,dt + K$$

But we can drop const K because
$$y = y_1 v$$
$$= c y_1 \int U(t)\,dt + Ky_1$$

Just a multiple of y_1 which is a known soln. NOT LI

cont'd →

p.100

Thus the 2 soln's are y_1 and $y_1 \cdot \int u\,dx$

we know $\int u\,dx \neq$ const because only $\int 0 =$ const.

$\Rightarrow$ these solns are LI

$\square$

p.100 example

Show $y(x) = x$ is soln of Legendre $\overset{n=1}{(1-x^2)\,y'' - 2xy' + 2y = 0}$ ⊛ $x \in (-1,1)$

and find a 2nd soln.

$y_1 = x$ Legendre

$y_1' = 1$ ⊛ $\Rightarrow (1-x^2)\cdot 0 - 2x\cdot 1 + 2x = 0$

$y_1'' = 0$ $\Rightarrow$ $0 = 0$ ✓ $y_1(x) = x$ is a soln.

Now find another soln: $y(x) := v(x)\overset{y_1}{x}$

$y' = xv' + v$

$y'' = v' + xv'' + v' = xv'' + 2v'$

Subs into Legendre: