Avez DC ch 7.4.1
p. 90–91 certain parts only

9/10/2023

Open $\mathcal{U} \subseteq E$ Banach $\quad \dot{x} = X(x)$ in $\mathcal{U}$ that is $\dot{\varphi}_x(t) = X(\varphi_x(t))$ in L&S ch 9, notation

**Def** $h : \mathcal{U} \to \mathbb{R}$ is a '<u>First Integral of $X$</u>' or '<u>Conserved Quantity</u>' or '<u>Const of motion</u>' if $\forall$ integral curves $\varphi_x : I_x \to \mathcal{U}$ we have $h \circ \varphi_x$ does not depend on $t$

i.e $h(\varphi_x(t)) = \text{const} \ \forall t \in I_x$ (the value of const may depend on base pt $x$)

i.e. $\frac{d}{dt}(h \circ \varphi_x) = 0$

**Lemma 7.2** a $C^1$ smooth fcn $h : \mathcal{U} \to \mathbb{R}$ is a 1st Integral of $X$ $\iff$ $Dh_x(X(x)) = 0 \ \forall x \in \mathcal{U}$

This is Lie deriv of $h$ along flow of $X$

Remark from L&S ch 9.6 (my writeup sheet ⑤) Lie derivative $(Xh)(x) \equiv (\mathcal{L}_X h)(x) := (h \circ \varphi_x)'(0)$

$$= Dh_{\varphi(0,x)}(\dot{\varphi}_x(0))$$
$$= Dh_x(X(x))$$

Thus a first integral has Lie deriv $= 0$ along streamlines of $X$.

**Pf** ($\Rightarrow$) we have $h(\varphi_x(t)) = c \quad \forall t$

Thus $(h \circ \varphi_x)'(t) = 0 \quad \forall t$

But $(h \circ \varphi_x)'(t) = Dh_x(X(x))$ ✓

($\Leftarrow$) Fix arb $x_0 \in \mathcal{U}$. $\exists!$ flow curve $\varphi_{x_0} : I_{x_0} \to \mathcal{U}$ Let $I_{x_0}$ be maximal length

Want to show that $h$ is const on this curve.

Let $\psi(t) := h(\varphi_{x_0}(t))$ and say $x = \varphi_t(x_0) = \varphi_{x_0}(t)$

$\psi'(t) = Dh_{\varphi_{x_0}(t)}(\varphi_{x_0}'(t)) = Dh_x(X(x)) = 0$ by hypoth

$\Rightarrow \psi'(t) = 0 \Rightarrow \psi(t) = \text{const} = h(\varphi_{x_0}(t))$

Is $h(\varphi_{x_0}(t))$ the <u>same</u> const for all pts on $\varphi_{x_0}(I_{x_0})$? Yes, because interval $I_{x_0}$ is conn, $\varphi$ is cont fcn, thus $\varphi(I)$ conn. $\square$

example p.91 $\quad X : \mathbb{R}^3 \to \mathbb{R}^3$, $\quad x \mapsto \begin{bmatrix} y-z \\ z-x \\ x-y \end{bmatrix}$ has $h(x) = x^2+y^2+z^2$ as a 1st Integral or Const of motion:

$$Dh_x(X(x)) = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix} \begin{bmatrix} y-z \\ z-x \\ x-y \end{bmatrix} = 2x(y-z) + 2y(z-x) + 2z(x-y) = 0 \ ✓$$

By the notation of L&S ch 9.8 (see my writeup sheets)

we could write $X =: (y-z)\frac{\partial}{\partial x} + (z-x)\frac{\partial}{\partial y} + (x-y)\frac{\partial}{\partial z}$

Then Lie Deriv is written $Xh = (y-z)\frac{\partial h}{\partial x} + (z-x)\frac{\partial h}{\partial y} + (x-y)\frac{\partial h}{\partial z}$

$$= (y-z)2x + (z-x)2y + (x-y)2z$$

and this is the same as above.

$\square$

Avez DC ch 8
p. 107

spivak DG v.1 p. 205

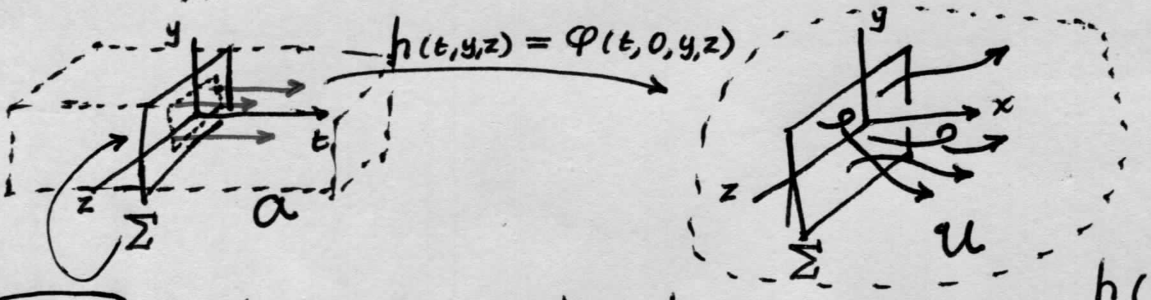
9/20/2023

# Vf Local Straightening Thm (Flow box Thm)

**Warm-up** Let $X : \mathcal{U} \to \mathbb{R}^3$ smooth vf, $\mathcal{U} \subset \mathbb{R}^3$
$X(0) \neq \vec{0}$ say $X(0) = \hat{e}_1$
$\Rightarrow$ $\exists$ a COV $h \ni$ in a nbhd of $0$
$X_h(x) = \hat{e}_1$
and the flow is a straight line
flow $\sigma_t(x) = x + t\hat{e}_1$

Pf. we know $\dot{\varphi}_x(t) = X(\varphi_x(t))$

**Step 1** Take a planar slice thru the origin $\perp$ to $X(0) = \hat{e}_1$ – Thus $y,z$ plane. (Think of Poincaré section $\Sigma$). We will make a COV that replaces the $x$ co-ord with the time $t$.




when $t=0$
$h\big|_{\Sigma} : \Sigma \to \Sigma$ id
$\varphi_0(0,y,z) = \begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$
$h(0,0,0) = \langle 0,0,0 \rangle$ ($t\ y\ z$ ; $x\ y\ z$)

**Step 2** Use Inv Fcn Thm to show $h$ is a diffeo in a nbhd of $\langle 0,0,0\rangle$:

$$Dh_{tyz} = \begin{bmatrix} \frac{\partial\varphi}{\partial t} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{bmatrix} \qquad Dh_{000} = \begin{bmatrix} X(0) & \begin{matrix} 0 \\ I \end{matrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\dot{\varphi}_x(t) = X(\varphi_x(t))$

$\varphi_{t=0} : \mathbb{R}^3 \to \mathbb{R}^3$ is Id.

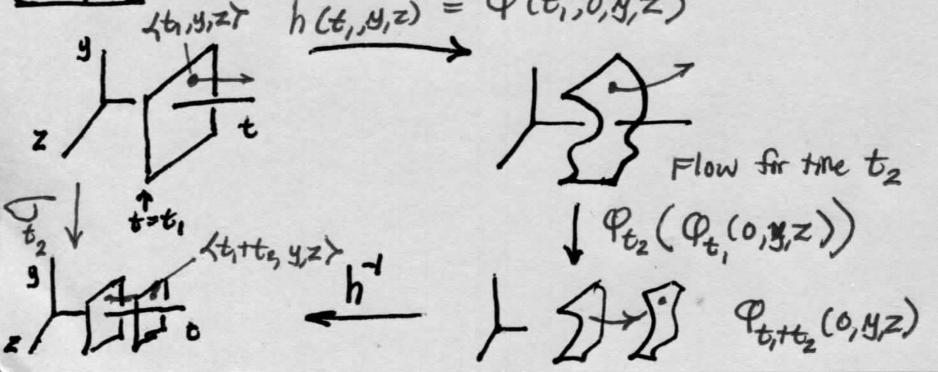
clearly nsing

So by cutting down the size of the nbhd $a$ around the origin (I am just going to leave the pictures and notation the same) we have that $h$ is a diffeo, and for $t$ small enough, $h(\{t\} \times \Sigma)$ stays in $\mathcal{U}$ (there isn't enough time for it to flow out).

▷ How do we know we can form a co-ord patch $(-\varepsilon, \varepsilon) \times \Sigma$ ? In other words, what if we can't find a pos $t = \varepsilon > 0$ that works $\forall$ pts in $\Sigma$?

We know we can: Because there is an open set containing $0$ in $a$, we know $\exists$ a small clsd rectangle in $\Sigma$ containing $0$. This we will call $\Sigma$. $\Sigma$ is cpt and the distance each pt flows is a cont fcn, thus this fcn attains a max on $\Sigma$ so we can bound the distance any pt will flow. So we can choose $t = \varepsilon$ small enough that $(-\varepsilon, \varepsilon) \times \Sigma$ is contained in $\mathcal{U}$ and the co-ord patch is well defined.

**Step 3** Show that the flow is, in fact, straight lines in $a$




Thus in the $\langle t,y,z\rangle$ co-ord sys we see the pt $\langle t_1, y, z\rangle$ flows to $\langle t_1+t_2, y, z\rangle$. This is a straight horiz line. $\sigma_{t_2}(p) = p + t_2\hat{e}_1$

[End of $\mathbb{R}^3$ illustrative warm-up]

cont'd →

Avez DC ch8 Straightening thm cont'd. p.107–108

Spivak DG vol 1 p.205
A&P AIDS p.26
A&M 2nd p.67

Meyer & Hall p.122
Symplectic Version

9/21/2023

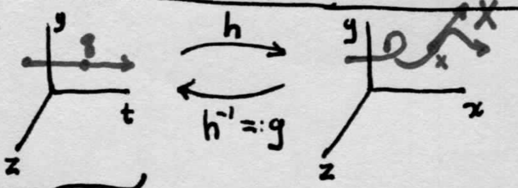
**Thm** $X$ smooth vf on mfd $M$, $X(\underset{\sim}{x_0}) \neq 0$ $\Big\} \Rightarrow$ $\exists$ a chart $(\alpha, U) \ni$ downstairs $X_\alpha(x) = \hat{e}_1 \quad \forall x \in \mathcal{a} \quad (\alpha(U) = \mathcal{a})$
and flow of $X_\alpha$ is $\varphi_t(x) = x + te_1$

Pf. We are just going to reduce this to the warm-up case – although $\mathbb{R}^n$ rather than $\mathbb{R}^3$.
We know $\exists$ a general chart $(\alpha, U)$ and we could compose $\alpha$ with a translation so $\alpha(\underset{\sim}{x_0}) = 0$ and a rotation and re-scaling so $X_\alpha(0) = \hat{e}_1$.
Now just apply the warm-up Thm

$\square$

**COR** $X$ smooth vf on $\mathbb{R}^n$, $X(x_0) \neq 0$ $\Big\} \Rightarrow$ $X$ has $n-1$ 1st Integrals (consts of motion) <u>locally</u> in a nbhd of $x_0$. Theirs derivs are LI.

Pf.




In this co-ord sys, the co-ord fcn $y, z$ are const on stream lines

$h(\xi) = x \qquad Dh_\xi(e_1) = X(h(\xi))$
$Dh_\xi^{①} = \dot{\varphi}_{h(\xi)}$

Let $g := h^{-1}$. We know $h^{-1}$ is a diffeo

$D(h^{-1})_x = \begin{bmatrix} | & & | \\ Dg_x^1 & \cdots & Dg_x^n \\ | & & | \end{bmatrix}$ These cols must be indep (LI)

It is obvious that $g^{ⓘ}$ for $i = 2, \dots, n$ are const on flow lines and thus are the 1st Integrals we seek. But let us show: $Dg_x^{ⓘ}(X(x)) = 0$
(p.90 Lie deriv $\equiv 0$)

$$Dg_x^{ⓘ}(X(x)) = D(\pi_i \circ g)_x(X(x)) = D\pi_i\, Dg_x(X(x)) = \pi_i\, Dg_{h(\xi)}(Dh_\xi(e_1))$$
$$= \pi_i\, D(h^{-1})_{h(\xi)} Dh_\xi(e_1) = \pi_i(e_1) = 0 \text{ since } i \neq 1$$

$\square$

p.109

$\triangleright$ Here is a sketch of a counterexample to show the 1st Integrals need not be defined globally, even if $X$ is $C^\infty$ and no points $X = 0$

Consider $\mathbb{R}^4$. Let pt $x = \langle q_1\, q_2,\, p_1\, p_2 \rangle$

$$X(x) = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & & & \\ & -2 & & 0 \end{array}\right] \begin{bmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ -q_1 \\ -2q_2 \end{bmatrix}$$

$X \neq 0$ everywhere in $\mathcal{U} = \mathbb{R}^4 - \{0\}$

$\overset{4}{n} - 1 = 3$ so there should be 3 1st Integrals. Here are 2 of them:

$f : \mathcal{U} \to R$
$x \mapsto p_1^2 + q_1^2$

$g : \mathcal{U} \to R$
$x \mapsto p_2^2 + 2q_2^2$

$Df_x = [2q_1\ 0\ 2p_1\ 0]$ obviously these are LI on $\mathbb{R}^4 - \{0\}$.
$Dg_x = [0\ 4q_2\ 0\ 2p_2]$

But there does not exist a 3rd one, because the flow curve of $X$, from an arb pt $a \in \mathcal{U}$, is everywhere dense in $f^{-1}(a) \cap g^{-1}(a)$. Showing this involves Jacobi–Kronecker Thm
Arnold & Avez p.115