Given a composite map $f\circ g$, we can compute $D(f\circ g)_x(\cdot) = \underbrace{Df_{g(x)}Dg_x(\cdot)}$

$$X \xrightarrow{g} Y \xrightarrow{f} Z \qquad x \longmapsto g(x) \longmapsto f(g(x)) \qquad h$$

How do we compute $\boxed{D^2(f\circ g)_x(\cdot,\cdot)} = D(Df_{g(x)}Dg_x)$?

we have Leibniz bilinear prod rule: $f\star g : E \longrightarrow F_1\times F_2 \longrightarrow G$ 'b' is the '$\star$'

$$x \longmapsto \langle f(x), g(x)\rangle \longmapsto b(f(x), g(x))$$

Then
$$D(f\star g)_x(h) = b(Df_x(h), g(x)) + b(f(x), Dg_x(h))$$
$$= Df_x(h)\star g(x) + f(x)\star Dg_x(h)$$

Now let '$\star$' be matrix multiplication - or more generally, one object operating linearly on another.
Think $A\vec{q}$ i.e. $A(x)\vec{q}(x) = A\star q$

Leibniz: $D(A\star q)_x(h) = DA_x(h)\star q(x) + A(x)\star Dq_x(h)$

Now let A depend on a fcn of x: $A(p(x))\,q(x) = (A\circ p)(x)\star q(x)$

$$D((A\circ p)\star q)_x(h) = D(A\circ p)_x(h)\star q + (A\circ p)\star Dq_x(h)$$
$$= \overbrace{DA_{p(x)}DP_x(h)}\star q + A\circ p(x)\star Dq_x(h)$$

now let $(A\circ p)(x) = Df_{g(x)}(\cdot)$

$q(x) = G^h(x) = Dg_x(h)$

$$D(Df_{g(x)}\star Dg_x(h))_x(k) = D(Df_{g(x)})_{g(x)}Dg_x(k)\star Dg_x(h) + Df_{g(x)}(\underbrace{DG^h_x(k)}_{D^2g_x(h,k)\text{ by def}})$$

$$D^2(f\circ g)_x(h,k) = D^2f_{g(x)}(Dg_x(k), Dg_x(h)) + Df_{g(x)}(D^2g_x(h,k))$$

It doesn't matter if I got order wrong, since D^2f is sym.

Avez only gives an unintelligible formula that applies for any D^n

Let's compute some concrete cases in $\mathbb{R}^2$ and $\mathbb{R}^3$ and see if we can build up to this:

1 $A(t)q(t) = [a(t)\ b(t)\ c(t)]\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \sum^3 a_i(t)q_i(t)$ which we can easily do by ordinary calculus:

row vector

$$\frac{d}{dt}(Aq) = \sum(\dot{a}_i q_i + a_i\dot{q}_i)$$

$$= [\dot{a}\ \dot{b}\ \dot{c}]\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + [a\ b\ c]\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

Thus for $A\star q : \mathbb{R} \longrightarrow (\mathbb{R}^3)^*\times\mathbb{R}^3 \longrightarrow \mathbb{R}$

$$t \longmapsto \langle A(t), q(t)\rangle \longmapsto b(A,q) = [a\,b\,c]\begin{bmatrix} q \\ q \\ q \end{bmatrix}$$

$$D(Aq)_t(h) = h[\dot{a}\ \dot{b}\ \dot{c}]\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + [a\ b\ c]\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}h$$

(2) Now consider $A(x,y)\, q(x,y)$

$$H := \begin{bmatrix} a(x,y) & b(x,y) & c(x,y) \end{bmatrix} \begin{bmatrix} q_1(x,y) \\ q_2 \\ q_3 \end{bmatrix} = \sum a_i(x,y)\, q_i(x,y)$$

$x = \langle x, y \rangle$

$$DH_x = \begin{bmatrix} \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} \end{bmatrix}$$ ← actually, since A was a row vector, this will be different (transposed) $[\cdot\ \cdot]^T$

$$H_x = \frac{\partial a}{\partial x} q_1 + a \frac{\partial q_1}{\partial x} + b_x q^2 + b q^2_x + c_x q^3 + c q^3_x$$

$$= \begin{bmatrix} a_x & b_x & c_x \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} q^1_x \\ q^2_x \\ q^3_x \end{bmatrix}$$

$$H_y = a_y q_1 + a q^1_y + b_y q^2 + b q^2_y + c_y q^3 + c q^3_y = \begin{bmatrix} a_y & b_y & c_y \end{bmatrix} \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} + \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} q^1_y \\ q^2_y \\ q^3_y \end{bmatrix}$$

Then

$$\begin{bmatrix} H_x & H_y \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} h_1 + \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} q^1_x \\ q^2_x \\ q^3_x \end{bmatrix} h_1 + \begin{bmatrix} a_y & b_y & c_y \end{bmatrix} \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} h_2 + \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} q^1_y \\ q^2_y \\ q^3_y \end{bmatrix} h_2$$

transposed from what you might have expected →

$$D(A * q)_x(h) = \underbrace{\begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix}}_{DA_x(h) * q} \begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} + \underbrace{\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} q^1_x & q^1_y \\ q^2_x & q^2_y \\ q^3_x & q^3_y \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}_{A * Dq_x(h)}$$

(3) Now $A(p(x,y)) * \vec{q}(x,y)$

$$H = \begin{bmatrix} a(p^1(x,y), p^2(x,y)) & b(p^1, p^2) & c(p^1, p^2) \end{bmatrix} \begin{bmatrix} q^1(x) \\ q^2(x) \\ q^3(x) \end{bmatrix} = \sum a_i(p(x,y))\, q^i(x,y)$$

$$H_x = \Big(\underbrace{\frac{\partial a}{\partial p_1}}_{D_1 a} p^1_x + \underbrace{\frac{\partial a}{\partial p^2}}_{D_2 a} p^2_x\Big) q^1 + a q^1_x + (b_{p^1} p^1_x + b_{p^2} p^2_x) q^2 + b q^2_x + (c_{p^1} p^1_x + c_{p^2} p^2_x) q^3 + c q^3_x$$

abbrev

$$H_y = (a_1 p^1_y + a_2 p^2_y) q^1 + a q^1_y + (b_1 p^1_y + b_2 p^2_y) q^2 + b q^2_y + (c_1 p^1_y + c_2 p^2_y) q^3 + c q^3_y$$

$$DH_x(h) = \begin{bmatrix} H_x & H_y \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} =$$

$$h_1 \Big[(a_1 p^1_x + a_2 p^2_x) q^1 + (b_1 p^1_x + b_2 p^2_x) q^2 + (c_1 p^1_x + c_2 p^2_x) q^3 + a q^1_x + b q^2_x + c q^3_x \Big] +$$

$$+ h_2 \Big[(a_1 p^1_y + a_2 p^2_y) q^1 + (b_1 p^1_y + b_2 p^2_y) q^2 + (c_1 p^1_y + c_2 p^2_y) q^3 + a q^1_y + b q^2_y + c q^3_y \Big]$$

$$\to \begin{bmatrix} p^1_x & p^2_x \\ p^1_y & p^2_y \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} q^1$$

cont'd →

Case 3 cont'd

$$D(\underbrace{A\circ p * g}_{H})_x(h) = [h^1\ h^2]\begin{bmatrix} p^1_x & p^2_x \\ p^1_y & p^2_y \end{bmatrix}\begin{bmatrix} D_1 a & D_1 b & D_1 c \\ D_2 a & D_2 b & D_2 c \end{bmatrix}\begin{bmatrix} g^1 \\ g^2 \\ g^3 \end{bmatrix} + [a\ b\ c]\begin{bmatrix} g^1_x & g^1_y \\ g^2_x & g^2_y \\ g^3_x & g^3_y \end{bmatrix}\begin{bmatrix} h^1 \\ h^2 \end{bmatrix}$$

$$D(A\circ p) * g \qquad\qquad A\, Dg_x(h)$$

$$h^T [Dg_x]^T [DA]^T g$$

4 Now bringing this all back to $Df_{g(x)} Dg_x(h)$:

$$Df_{g(x)} Dg_x(h) =: [a\circ p(x,y)\ \ b\circ p(x,y)\ \ c\circ p(x,y)]\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

we really were specifying:

$$f: \mathbb{R}^3 \to \mathbb{R}_3$$

$$g = p: \mathbb{R}^2 \to \mathbb{R}^3$$

$$Dp_x(h) = \begin{bmatrix} p^1_x & p^1_y \\ p^2_x & p^2_y \\ p^3_x & p^3_y \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} =: g(x) \qquad g: \mathbb{R}^2 \to \mathbb{R}^3$$

$$g(x) = G^h(x) = Dg_x(h)$$

so properly we would say $Dg_x(k)$

and $Dg_x(k) = D^2 g_x(h,k)$

$$\begin{bmatrix} \boxed{g^1_x\ \ g^1_y} \\ g^2_x\ \ g^2_y \\ g^3_x\ \ g^3_y \end{bmatrix}\begin{bmatrix} h^1 \\ h^2 \end{bmatrix}$$

To compute $D(Dg_x(h))(k)$ it is a vertical stack of the objects we considered in case 2

Then the result of 3 would properly be rewritten as:

$$DH_x(k) = [k_1\ k_2]\begin{bmatrix} p^1_x & p^2_x & p^3_x \\ p^1_y & p^2_y & p^3_y \end{bmatrix}\begin{bmatrix} D_1 a & D_1 b & D_1 c \\ D_2 a & D_2 b & D_2 c \\ D_3 a & D_3 b & D_3 c \end{bmatrix}\begin{bmatrix} g^1 \\ g^2 \\ g^3 \end{bmatrix} + [a\ b\ c]\underbrace{\begin{bmatrix} g^1_x & g^1_y \\ g^2_x & g^2_y \\ g^3_x & g^3_y \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}}_{Dg_x(k)}$$

Then to rewrite $Dg_x(k)$ in terms of g, we expand it as a vertical stack:

$$\begin{bmatrix} [k_1\ k_2]\begin{bmatrix} g^{1}_{xx} & g^{1}_{xy} \\ g^{1}_{yx} & g^{1}_{yy} \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ [k_1\ k_2]\begin{bmatrix} g^2_{xx} & g^2_{xy} \\ g^2_{yx} & g^2_{yy} \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ [k_1\ k_2]\begin{bmatrix} g^3_{xx} & g^3_{xy} \\ g^3_{yx} & g^3_{yy} \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \end{bmatrix}$$

Then at last

$$DH_x(k) \equiv D^2(f\circ g)_x(h,k) = k^T (Dg_x)^T D^2 f_{g(x)} Dg_x h + Df_{g(x)}\begin{bmatrix} k^T D^2 g^{1}_x h \\ k^T D^2 g^{2}_x h \\ k^T D^2 g^{3}_x h \end{bmatrix}$$

$$D(Df_{g(x)} Dg_x(h))(k)$$

$$[Dg_x(k)]^T D^2 f_{g(x)} Dg_x(h) + Df_{g(x)} D^2 g_x(h,k)$$

This serves to illustrate a concrete case, and plausibly confirm the formula

$$\boxed{D^2(f\circ g)_x(h,k) = D^2 f_{g(x)}(Dg_x(k), Dg_x(h)) + Df_{g(x)}(D^2 g_x(h,k))} \quad \square$$

p. 39 **Thm 4.5** $\left.\begin{array}{l} J : GL_{EF} \longrightarrow GL_{EF} \\ \quad u \longmapsto u^{-1} \end{array}\right\} \Rightarrow \begin{array}{l} J \text{ is } C^\infty \text{ smooth} \\ DJ_u(h) = -u^{-1} \cdot h \cdot u^{-1} \end{array}$ also show $D^2 J_u$

Pf. Step 1 we need to (a) derive the guess that $DJ_u(h) = -u^{-1} \cdot h \cdot u$ and then (b) verify it is Frechet deriv.

Recapping ~~From~~ Avez Lemma 3.2 p.25 $(u+h) = u[I + u^{-1}h]$ so $(u+h)^{-1} = \underbrace{[I + u^{-1}h]^{-1}}_{[I-N]^{-1} \text{ where } N = -u^{-1}h} u^{-1}$

Neumann Geo series $= \sum^{\infty} N^k = \sum (-u^{-1}h)^k$

then $J(u+\varepsilon h) = [I + \varepsilon u^{-1}h]^{-1} u^{-1}$

$$= \sum_{k=0}^{\infty} (-1)^k (u^{-1}h)^k u^{-1} \varepsilon^k = u^{-1} - u^{-1}hu^{-1}\varepsilon + (u^{-1}h)^2 u^{-1}\varepsilon^2 + -(u^{-1}h)^3 u^{-1}\varepsilon^3 + \dots$$

$$\frac{\partial J}{\partial \varepsilon} = -u^{-1}hu^{-1} + 2(u^{-1}h)^2 u^{-1}\varepsilon - 3(u^{-1}h)^3 u^{-1}\varepsilon^2 + \dots$$

$\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} = -u^{-1}hu^{-1}$ and recall from ch 1 first pages $DJ_u(h) = \left.\frac{d}{d\varepsilon} J(u+\varepsilon h)\right|_{\varepsilon=0}$ if LHS exists (Frechet)

(b) $J(u+h) - J(u) - L(h) = (u+h)^{-1} - u^{-1} + {}^{+}u^{-1} \cdot h \cdot u^{-1}$

$$= (u+h)^{-1}\left[I - (u+h)u^{-1} + (u+h)u^{-1}hu^{-1}\right]$$

$$\cancel{I} - \cancel{uu^{-1}} - \cancel{hu^{-1}} + \cancel{uu^{-1}}\cancel{hu^{-1}} + hu^{-1}hu^{-1}$$

$$= (u+h)^{-1}[hu^{-1}hu^{-1}]$$

then $\frac{\|J(u+h) - J_u - L(h)\|}{\|h\|} = \frac{\|(u+h)^{-1}[hu^{-1}hu^{-1}]\|}{\|h\|} \le \frac{|(u+h)^{-1}|\,\cancel{|h|}\,|u^{-1}|\,|h|\,|u^{-1}|}{\cancel{|h|}}$

$$= \underbrace{|(u+h)^{-1}|}_{\substack{\text{This goes to } |u^{-1}| \\ \text{since } J \text{ is cont} \\ \text{(by Avez Lemma 3.2)}}} |u^{-1}|^2 |h| \longrightarrow 0 \text{ as } |h| \to 0$$

Step 2 Now we must show J is infact C^∞

we just found $DJ_u(h) = -u^{-1}hu^{-1}$

identity map, with a doubling

Observe this is $DJ_{(\cdot)}(h) : u \xmapsto{J} u^{-1} \xmapsto{\Xi} (u^{-1}, u^{-1}) \xmapsto{b^h} -u^{-1}hu^{-1}$

Lets consider the map b^h more generally: $b^h(x,y) = -xhy$ This is bi linear in x and y; h is a param

is it Cont (ie Bdd Linear)?

$|b^h(x,y)| = |xhy| \le |x||h||y|$ so $\|b^h\|_{op} = \sup_{\substack{|x|=1 \\ |y|=1}} |b^h(x,y)| \le 1|h| \cdot 1$ For any given h, this is finite, so b^h Bdd Linear in x, y

and we know that a continuous bilinear map is actually C^∞.

Thus $DJ_u(h) = (\underbrace{b^h \circ \Xi}_{C^\infty} \circ J)(u)$

Here is the "weird" bootstrapping arg: we know J on RHS is C^0 (and Ξ, b^h C^∞). Thus LHS DJ_u is $C^0 \Rightarrow J$ on RHS is actually C^1

Repeat arg inductively $J\ C^1 \Rightarrow DJ\ C^1 \Rightarrow J\ C^2$ .... $\Rightarrow J$ is C^∞ $\square$

So we <u>don't</u> try to actually compute $D^k J_u$ and show it is Cont

The discussion continues →

Avez DC ch 4 $D^2 J_u(h,k) = u^{-1}hu^{-1}ku^{-1} + u^{-1}ku^{-1}$ 13/10/2024

p. 39 (9)

Thm 4.5 (J is C^∞) discussion:

I initially tried to compute $D^2 J = D(DJ) = D(b \circ \Phi \circ J) = Db_{\Phi J(u)} \, D\Phi_{J(u)} \, DJ_u$

I got tangled up doing this, so I will give the correct steps.

> Note: Avez p. 40 Gives this formula $D^k J_u(h_1, \dots, h_k) = \sum\limits_{\substack{\text{all perms} \\ \sigma}} (-1)^k (u^{-1} h_{\sigma(1)} u^{-1}) \circ \dots \circ (u^{-1} h_{\sigma(k)} u^{-1})$
> I think this formula is wrong! See top margin

On prev sheet $J(u + \varepsilon h) = [I + \varepsilon u^{-1} h]^{-1} u^{-1} = \sum (-1)^k (u^{-1}h)^k \varepsilon^k$

$u + th + sk$

$h \rightsquigarrow th + sk$

$(u + h) = u[I + u^{-1}h]$

$(u + (th + sk)) = u[I + u^{-1}(th + sk)]$

First, find what the value of $D^2 J_u$ should be by applying the Gateaux variation

$$\underbrace{(u + (th + sk))^{-1}}_{J} = [I - v]^{-1} u^{-1} = \sum_{\ell=0}^{\infty} (-1)^\ell [u^{-1}(th+sk)]^\ell u^{-1}$$

$$= u^{-1} - u^{-1}(th + sk)u^{-1} + [u^{-1}(th+sk)]^2 u^{-1} - [u^{-1}(th+sk)]^3 u^{-1} + \dots$$

For convenience of writing this out, lets abbreviate $p := u^{-1}$ $\quad \Omega := (th + sk)$

$$J = p - p\Omega p + p\Omega p\Omega p - p\Omega p\Omega p\Omega p + \dots$$

$\dot{\Omega} = h$

$$\frac{\partial J}{\partial t} = 0 - p\dot{\Omega}p + p\dot{\Omega}p\Omega p + p\Omega p\dot{\Omega}p - [p\dot{\Omega}p\Omega p\Omega p + p\Omega p\dot{\Omega}p\Omega p + p\Omega p\Omega p\dot{\Omega}p] + \dots$$

$$= -php + php\Omega p + p\Omega php - [php\Omega p\Omega p + p\Omega php\Omega p + p\Omega p\Omega php] + \dots$$

$\Omega' = k$

$$\frac{\partial}{\partial s}\frac{\partial J}{\partial t} = php\Omega'p + p\Omega'php - [php\Omega'p\Omega p + php\Omega p\Omega'p + p\Omega'php\Omega p + p\Omega php\Omega'p + p\Omega'p\Omega php + p\Omega p\Omega'php]$$

$$\left.\frac{\partial}{\partial s}\frac{\partial J}{\partial t}\right|_{\substack{t=0 \\ s=0}} = phpkp + pkphp - [0 + 0 + 0 + 0 + 0 + 0] + 0 \dots$$

$\Omega(0,0) = 0$

Thus

$$\delta^2 J_u(h,k) = \underbrace{u^{-1}hu^{-1}ku^{-1}}_{-b^k(u^{-1}hu^{-1},\, u^{-1})} + \underbrace{u^{-1}ku^{-1}hu^{-1}}_{-b^k(u^{-1},\, u^{-1}hu^{-1})}$$

where $b^g(x,y) = -xgy$ from prev sheet

2nd, we must get this value working with $DJ_u(h) = b^h(u^{-1}, u^{-1})$.

$$Db^g_{x_1 x_2}(h_1, h_2) = b^g(x_1, h_2) + b^g(h_1, x_2)$$

$$= -x_1 g h_2 + -h_1 g x_2$$

So

$$Db^k_{u^{-1}u^{-1}}(u^{-1}hu^{-1}, u^{-1}hu^{-1}) = b^k(u^{-1}hu^{-1}, u^{-1}) + b^k(u^{-1}, u^{-1}hu^{-1})$$

$$= u^{-1}hu^{-1}ku^{-1} + u^{-1}ku^{-1}hu^{-1}$$

cont'd →

Avez DC ch 4

Thm 4.5 (J is C^∞) discussion, cont'd:

$$
\begin{aligned}
D^2J_u(h,k) &= D\left(DJ^h\right)_u(k) \\
&= D\left(b^h \circ \mp \circ J\right)_u(k) \\
&= Db_{\mp J(u)}\left(D\mp_{J(u)}\left(DJ_u(h)\right)\right)(k) \\
&= Db_{(u^{-1}u^{-1})}\big(\underbrace{DJ_u(h)}_{-u^{-1}hu^{-1}}, \underbrace{DJ_u(h)}_{-u^{-1}hu^{-1}}\big)\ (k) \\
&= Db^k_{u^{-1}u^{-1}}\left(-u^{-1}hu^{-1}, -u^{-1}hu^{-1}\right) \\
&= b^k\left(-u^{-1}hu^{-1}, u^{-1}\right) + b^k\left(u^{-1}, -u^{-1}hu^{-1}\right) \\
&= +u^{-1}hu^{-1}ku^{-1} + u^{-1}ku^{-1}hu^{-1}
\end{aligned}
$$

recall $b^k(x,y) = -xky$

$\boxed{\text{DONE}}$

ch 4.2.7 Inv of a C^n diffeo

Recall from ch 3

$$
\begin{array}{ccc} E & & F \\ \mathcal{U} & \overset{f}{\underset{f^{-1}}{\rightleftarrows}} & V \end{array}
\qquad E \xrightarrow{Df_x} F
$$

by def f diffeo $\Rightarrow \begin{cases} f \text{ One-to-One} \\ \quad \text{Onto} \\ \quad C^1 \text{ smooth} \\ f^{-1}\ C^1 \end{cases}$

It is also true that Df_x is an iso because

$f^{-1} \circ f = I_E$ and $f \circ f^{-1} = I_F$

$D(f^{-1})_{f(x)} Df_x = I_E \qquad Df_{\underset{x}{f^{-1}(v)}} D(f^{-1})_{\underset{f(x)}{v}} = I_F$

That is to say: $\exists$ linear map $A \ni A \cdot Df_x = I$ and $Df_x A = I$

A is left and right inv $\Rightarrow$ The inv $A = [Df_x]^{-1}$

In fact $[Df_x]^{-1} = D(f^{-1})_{f(x)}$

Thm 4.6 f C^n smooth diffeo $\Rightarrow f^{-1}$ C^n Smooth also

pf. we just showed $D(f^{-1})_{v=f(x)} = [Df_x]^{-1} = \left[Df_{f^{-1}(v)}\right]^{-1}$

$= J \circ Df \circ f^{-1}$

Pf by induct

we know it is true for $n=1$ by def of diffeo

assume $f \in C^{n-1} \Rightarrow f^{-1} \in C^{n-1}$. Now let $f \in C^n$

Then $D(f^{-1}) = \underset{C^\infty}{J} \circ \underset{C^{n-1}}{Df} \circ \underset{C^{n-1}}{f^{-1}} \Rightarrow D(f^{-1})$ is $C^{n-1} \Rightarrow f^{-1}\ C^n$

$\square$