Abraham, Marsden, Ratiu Mflds, Tensor Analysis, Applications

From ~~Marsden, Tudor, Ratiu~~

P. 119

The Size of the Nbhds in Inv FT

$x_0 \equiv a$ and $f(x_0) = y_0$

Prop 2.5.6 $f : \mathcal{U} \subseteq E \longrightarrow F$ C^r smooth $r \geq 2$

Df_a iso

Define $L := \|Df_a\|$, $M := \|Df_a^{-1}\|$

Let $\|D^2 f_x\| \leq K$ for $\|x - a\| \leq R$ and $B_R(a) \subseteq \mathcal{U}$

Define $P := \min\left(\frac{1}{2KM}, R\right)$

$Q := \min\left(\frac{1}{2NL}, \frac{P}{M}, P\right)$

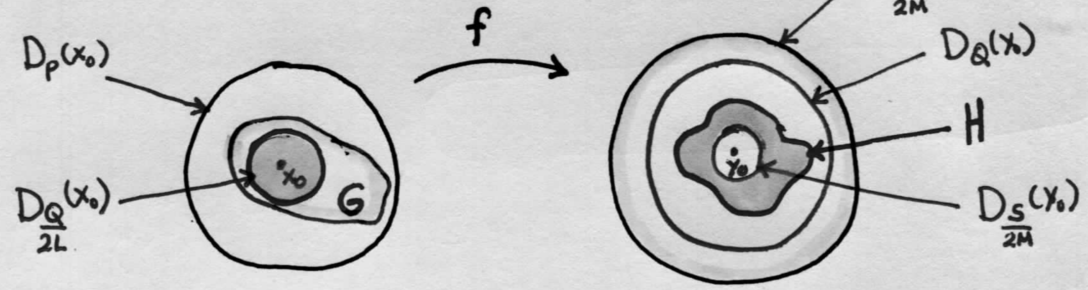
$S := \min\left(\frac{1}{2KM}, \frac{Q}{2L}, Q\right)$

$N := 8M^3K$

$\Longrightarrow$

$G \xrightarrow{f \text{ diffeo}} D_{\frac{P}{2M}}(y_0)$

$D_{\frac{Q}{2L}}(x_0) \xleftarrow{f^{-1} \text{ diffeo}} H$



The Corollary coming up is the real Imp Fcn Thm.

Thm 3.2 (thickening of usual Imp FT)
(Note domain is split)

$f: \mathcal{U} \times \mathcal{V} \xrightarrow{C^1} G$ (with $\mathcal{U}\times\mathcal{V} \subseteq E \times F$; E, F, G Banach spaces)

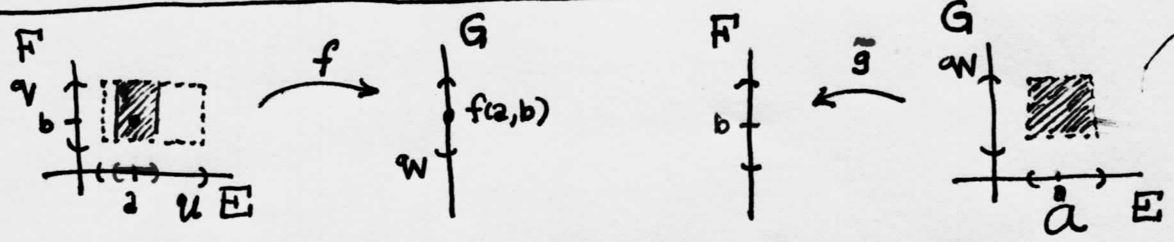
$D_2 f_{(a,b)} : F \longrightarrow G$ iso

$\Longrightarrow$ $\exists$ open sets $a \in \mathcal{A}$, $f(a,b) \in \mathcal{W}$

$\exists!\ \tilde{g}: \mathcal{A} \times \mathcal{W} \xrightarrow{C^1} \mathcal{V} \ni$

$f(x, \tilde{g}(x,w)) = w \quad \forall \langle x,w\rangle \in \mathcal{A} \times \mathcal{W}$

tilde because $\tilde{g}$ here depends on w



The shaded blue area is mapped to shaded blue area by φ

Pf: The whole idea is to construct a fcn $\varphi: \mathcal{U}\times\mathcal{V} \subseteq E \times F \longrightarrow E \times G$, $\langle x,y\rangle \longmapsto \langle x, f(x,y)\rangle$

and apply Inv FT to φ. $\varphi = \pi_1 \times f$

Step 1 Show that $D\varphi_{ab}(\langle h,k\rangle) = \langle h, D_1 f_{ab}(h) + D_2 f_{ab}(k)\rangle$

we need linear $L \ni \dfrac{\| \varphi(a+h, b+k) - \varphi(a,b) - L(\langle h,k\rangle)\|}{\|\langle h,k\rangle\|} \longrightarrow 0_{E\times G}$ as $\|\langle h,k\rangle\| \to 0$

$$\varphi(a+h, b+k) - \varphi(a,b) = \langle a+h, f(a+h, b+k)\rangle - \langle a, f(a,b)\rangle = \langle h, f(a+h, b+k) - f(a,b)\rangle$$

Hence we take $L(\langle h,k\rangle) := \langle h, Df_{ab}(\langle h,k\rangle)\rangle$

Then $\dfrac{\|\varphi(a+h, b+k) - \varphi(a,b) - L(\langle h,k\rangle)\|}{\|\langle h,k\rangle\|} = \left\| \left\langle 0, \dfrac{f(a+h,b+k) - f(a,b) - Df_{ab}(\langle h,k\rangle)}{\|\langle h,k\rangle\|} \right\rangle \right\|$

since f is difb, 2nd quantity $\longrightarrow 0$

And we know from ch 1 sheet (8) $Df_{ab}(\langle h,k\rangle) = D_1 f_{ab}(h) + D_2 f_{ab}(k)$

Step 2 Show $D\varphi_{ab}$ is invertible by explicitly exhibiting the inverse.

we need a linear map $Q \ni \begin{cases} Q \circ D\varphi_{ab}(\langle h,k\rangle) = \langle h,k\rangle \\ D\varphi_{ab} \circ Q(\langle h',k'\rangle) = \langle h',k'\rangle \text{ we don't use this one} \end{cases}$

we just work with 1st component,

$Q(\langle h, D_1 f_{ab}(h) + D_2 f_{ab}(k)\rangle) \overset{!}{=} \langle h,k\rangle \Rightarrow Q^{(1)}(\langle h',k'\rangle) = \pi_1(\langle h',k'\rangle) = h'$

For $Q^{(2)}$, note that if $D_1 f(h) + D_2 f(k) = k' \Rightarrow k = D_2 f^{-1}(k' - D_1 f(h))$

Thus $Q^{(2)}(\langle h',k'\rangle) = D_2 f_{ab}^{-1}(k' - D_1 f_{ab}(h'))$

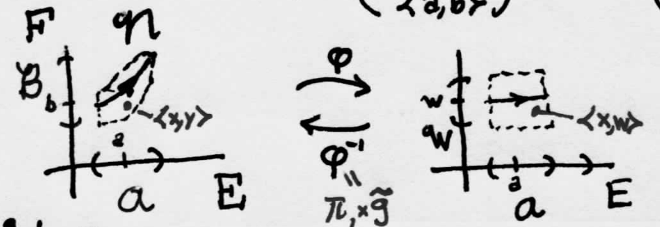
Explicit formula for $(D\varphi_{ab})^{-1} = Q(\langle h',k'\rangle) = \langle h', D_2 f_{ab}^{-1}(k' - D_1 f_{ab}(h'))\rangle$

cont'd →

Step 3 apply Inv FT to φ

$$\varphi : \begin{pmatrix} \text{nbhd of} \\ \langle a,b\rangle \end{pmatrix} \xrightarrow{\text{diffeo}} \begin{pmatrix} \text{nbhd of} \\ \varphi(a,b) \end{pmatrix}$$

($E \times F$ over the first neighborhood, $E \times G$ over the second) ← trimming down, we can write this $\mathcal{a} \times W$



For every $\langle x,w\rangle \in \mathcal{a} \times W$ $\exists$ pt $\langle x,y\rangle \in \mathcal{n}$ $\ni$ $\varphi^{-1}(\langle x,w\rangle) = \langle x,y\rangle$

i.e. $(\varphi^{-1})^{①}(x,w) = x$ Define $\tilde{g} := (\varphi^{-1})^{②}$ and clearly $\tilde{g}$ is smooth

$(\varphi^{-1})^{②}(x,w) = y$ $\quad f(x, \tilde{g}(x,w)) = w$

$\square$

COR 3.2 Standard Implicit Fcn Thm

Same basic hypotheses as before. Note how domain is split

$f : \mathcal{U} \times \mathcal{V} \xrightarrow{C^1} G$ (with $\mathcal{U} \times \mathcal{V} \subset E \times F$)

$D_2 f_{ab} : F \xrightarrow{\text{iso}} G$

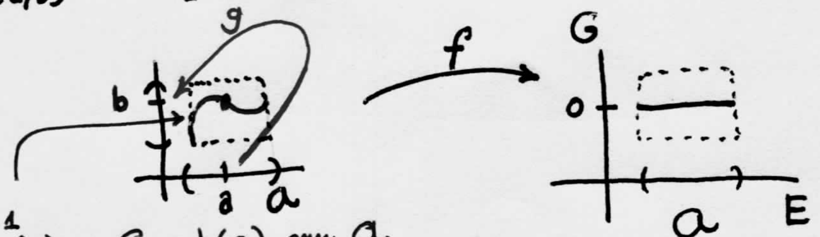
$f(a,b) = 0$ [so we restrict to $w=0$]

$\Longrightarrow$

① $\exists!\ g : \mathcal{a} \xrightarrow{C^1} \mathcal{B} \ni f(x,g(x)) = 0$ (with $\mathcal{a} \subset E$, $\mathcal{B} \subset F$)

that is: $f^{-1}(0) = \text{Graph}(g)$ over $\mathcal{a}$

② $Dg_x = -D_2 f^{-1}_{(x,g(x))}\left(D_1 f_{(x,g(x))}\right)$



$f^{-1}(0) = \text{Graph}(g)$ over $\mathcal{a}$

Pf. Take $g(x) := \tilde{g}(x,0)$ Thus we have $\tilde{g} : \mathcal{a} \times \{0\} \xrightarrow{C^1} \mathcal{V}$

so the range is $\mathcal{B} := \tilde{g}(\mathcal{a} \times \{0\})$

$\forall x \in \mathcal{a}$ $f(x, g(x)) = f(x, \tilde{g}(x,0)) = 0$ from Thm.

And since we have $f(x, g(x)) = 0$ apply $\frac{d}{dx}$ to both sides

$D_1 f_{(x,g(x))} + D_2 f_{(x,g(x))} Dg_x = 0 \Longrightarrow D_2 f_{(x,g(x))} Dg_x = -D_1 f_{(x,g(x))}$

But is $D_2 f_{(x,g(x))}$ invertible? We know $D_2 f_{(a,g(a))}$ is. $A := D_2 f_{(a,b)}$ (with $g(a) = b$)

On ch 3 sheet ③ we have the Thm that Invertible linear maps form an open set.

$\|A - B\| < \frac{1}{\|A^{-1}\|} \Rightarrow B^{-1}$ exists. We know $D_2 f_{(x,g(x))}$ is cont in x, so for x near a,

$D_2 f_{(x,g(x))}$ is near A and thus $D_2 f^{-1}_{(x,g(x))}$ exists

$$\Longrightarrow Dg_x = -D_2 f^{-1}_{(x,g(x))}\left(D_1 f_{(x,g(x))}\right)$$

$\square$

Avez DC ch3

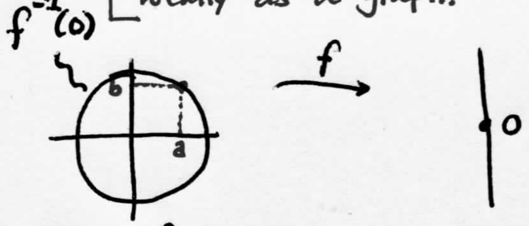
[Geometrically, Imp FT is a local Pre-image Thm (G&P), combined with representing a mfd locally as a graph.]

6/22/2024 (11)

Let's do some examples

domain splitting $f: \mathbb{R}^{1}_x \times \mathbb{R}^{1}_y \longrightarrow \mathbb{R}^{1}$

$\langle x, y\rangle \longmapsto x^2+y^2-1$



In this example, we don't need the Imp FT, we can explicitly solve algebraically:

$x^2+y^2-1=0$

$y^2 = 1-x^2$

$y_1(x) = \sqrt{1-x^2}$

$y_2(x) = -\sqrt{1-x^2}$

We could also differentiate implicitly (Imp FT says y' exists)

$2x + 2yy' = 0$

$y' = \frac{-x}{y}$ where $y \neq 0$

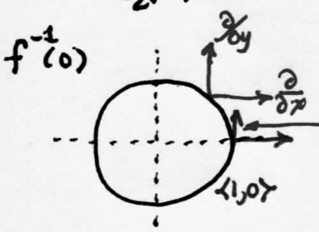
For $\langle a,b\rangle$ on the circle as shown, we have $f(a,b)=0$ and we want a fcn $y=y(x) \ni f(x,y(x))=0$

$D_2 f_{ab} = \frac{\partial}{\partial y}(x^2+y^2-1)\Big|_{ab} = 2y\Big|_{b} = 2b \neq 0$ for b as shown $(b\neq 0)$

In this example, we can illustrate the geometric meaning of this

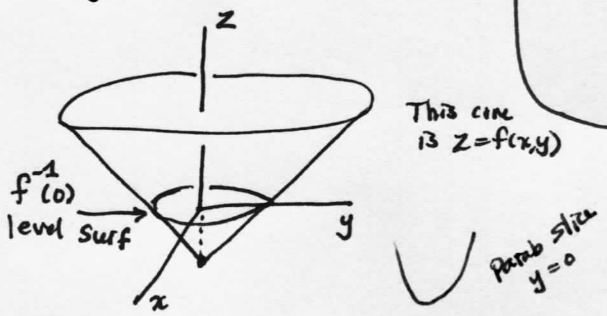
consider $z = f(x,y) = x^2+y^2-1$ the graph of f.

$D_2 f \neq 0$ is saying $\frac{\partial z}{\partial y} \neq 0$



$\frac{\partial}{\partial y}$ is tangent to level surf at $\langle 1,0\rangle$

$\Rightarrow \frac{\partial}{\partial y} f(1,0) = 0$



This $D_2 f_{ab} \neq 0$ means $Df_{ab} = [f_x \; f_y] \neq [0,0]$ so $f^{-1}(0)$ is locally a submfd in nbhd of $\langle a,b\rangle$

$f^{-1}(0)$ is a level curve of $z=f(x,y)$. At pt $\langle a,b\rangle$ z is incr in both the "$\frac{\partial}{\partial x}$" $= e_1$ and "$\frac{\partial}{\partial y}$" $= e_2$ directions. But at $\langle 1,0\rangle$ and $\langle -1,0\rangle$, $\frac{\partial}{\partial y}$ is tangent to the level curve $\to f$ is not changing in that dir Thus $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = D_2 f$ is 0 there.

$y(x)$ [a.k.a. $g(x)$] $\frac{dy}{dx} = \frac{-\partial z/\partial x}{\partial z/\partial y} = \frac{-\partial z/\partial x}{0}$ undefined (vertical slope)

[But at $\langle 1,0\rangle$, no problem with $x = x(y)$]

For $\langle x,y\rangle \neq \langle 1,0\rangle$ or $\langle -1,0\rangle$ the 2nd part of Imp FT gives

$$Dg_x = -D_2 f^{-1}(D_1 f) \quad \text{or} \quad \frac{dg}{dx} = -\left(\frac{\partial z}{\partial y}\right)^{-1}\left(\frac{\partial z}{\partial x}\right) = \frac{-1}{2y}2x = \frac{-x}{y}$$

[we could solve this ODE $y\,dy = -x\,dx \Rightarrow x^2+y^2=C$ but not telling us more than we know]

(ex 2) Linear version (could be considered a prototype) Rudin PDMA p.224

$A: \mathbb{R}^m \times \mathbb{R}^{(n)} \longrightarrow \mathbb{R}^{(n)}$ $\quad A(0,0)=0$

$\langle x,y\rangle \longmapsto A\begin{bmatrix} x \\ y\end{bmatrix}$

$$\left[\, A_1 \;\vdots\; A_2 \,\right]\begin{bmatrix} x \\ y\end{bmatrix} = \vec{0} \Rightarrow A_1 x + A_2 y = 0 \Rightarrow A_2 y = -A_1 x$$

If A_2^{-1} exists, we can solve $y = \underbrace{-A_2^{-1}(A_1 x)}_{y(x)}$

Then $A(x, y(x)) = 0 \;\; \forall x \in \mathbb{R}^m$ (global Imp FT)

$A^{-1}(0) = \ker(A)$ is a subsp of $\mathbb{R}^{m+n}$ i.e. submfd.
m-dim

Since $y(x)$ is linear fcn,

$Dy_x(h) = -A_2^{-1}A_1 h$

□

Avez DC ch3

6/9/2024

(12)

Example from M&T ch 4 ex 10 p. 242

Can we solve $x^2y^2 + xzu + yv^2 = 3$, $u^3yz + 2xv - u^2v^2 = 2$ near pt $\langle x,y,z,u,v\rangle = \langle 1,1,1,1,1\rangle$? call this a

Define $f: \mathbb{R}^3 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\langle \underbrace{x,y,z}_{x}, \underbrace{u,v}_{u} \rangle \longmapsto \begin{bmatrix} xy^2 + xzu + yv^2 - 3 \\ yzu^3 + 2xv - u^2v^2 - 2 \end{bmatrix}$$

We seek soln's on $f^{-1}(0)$, that is $\{ \text{all } \langle x,u\rangle \mid f(x,u) = 0 \}$

Can we apply Imp FT?

$$D_2 f_a = D_{\langle u,v\rangle} f = \begin{bmatrix} \frac{\partial f^{(1)}}{\partial u} & \frac{\partial f^{(1)}}{\partial v} \\ \frac{\partial f^{(2)}}{\partial u} & \frac{\partial f^{(2)}}{\partial v} \end{bmatrix} = \begin{bmatrix} xz & 2vy \\ (3u^2yz - 2uv^2) & (2x - 2vu^2) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

plug in $\langle 1,1,1,1,1\rangle$

$\det(D_2 f_a) = -2 \neq 0$ so $D_2 f_a$ is an iso

By Imp FT, $\exists\, g: \overset{\mathbb{R}^3}{a} \to \overset{\mathbb{R}^2}{B}$, $\langle x,y,z\rangle \longmapsto \begin{bmatrix} u(x,y,z) \\ v(x,y,z) \end{bmatrix}$

the fcn g exists, but we don't have an explicit form to write down.

(b) $Dg_x = -(D_2 f_{\langle x,g(x)\rangle})^{-1}(D_1 f_{\langle x,g(x)\rangle})$

Compute $D_1 f$ and evaluate at $\langle x,y,z\rangle = \langle 1,1,1\rangle$

$$D_1 f = \begin{bmatrix} (y^2 + zu) & (2yx + v^2) & (xu) \\ 2v & u^3 z & u^3 y \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$Dg_x \Big|_{\langle 1,1,1\rangle} = -\frac{1}{-2} \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix} = +\frac{1}{2} \begin{bmatrix} -4 & -2 & -2 \\ 0 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 & -1 \\ 0 & (-1) & 0 \end{bmatrix}$$

This is $\frac{\partial v}{\partial y}(1,1,1)$

The question specifically asked for $\frac{\partial v}{\partial y}(1,1,1)$

M&T VC ch 4.4

we could use implicit differentiation for a slightly shorter way for a specific result like this.

Thm 3.3 $f: \mathcal{U} \subseteq E \xrightarrow{C^1} F$ Banach spaces
$a \in \mathcal{U}$ open set
$E = \ker(Df_a) \oplus E_2$ where E_2 is a topologically clsd subsp (so Banach sp)
$Df_a : \ker(Df_a) \oplus E_2 \xrightarrow{\text{Onto}} F$
$\Rightarrow$ $f(\mathcal{U})$ contains an open nbhd of $f(a)$

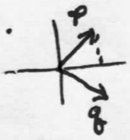
(Another COR of thickened Imp FT Thm 3.2)

Pf. $\ker(Df_a) = Df_a^{-1}(\{0\})$ Df_a cont, so $Df_a^{-1}(\{0\})$ is a topologically clsd set.
$\Rightarrow$ Its a Banach sp.

We need to meet the hypotheses of (Thm 3.2), so we can use it.

[Note that we don't need the hypothesis that E_2 is clsd if E is fin dim, or if E is Hilbert space, because then take $E_2 = (\ker(Df_a))^\perp$]

Remark: I wanted to say $E_2 := \{\text{all } h \in E \mid Df_a(h) \neq 0\}$ but this DOES NOT WORK.
Consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\ker(A) = \text{Span}\{e_1\}$ but everything not on x axis is NOT a valid subsp
$\vec{p} + \vec{q} \in \ker(A)$ so this candidate for E_2 is not even algebraically closed

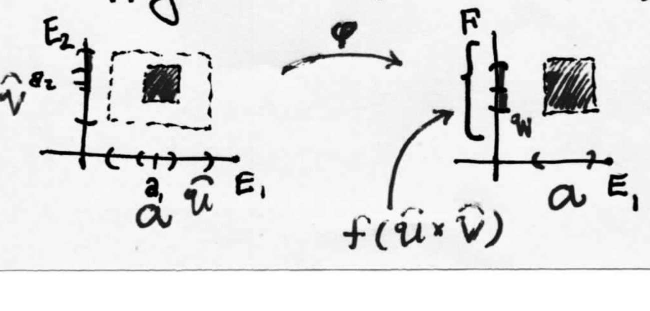


So $E = E_1 \oplus E_2$ means any $v \in E$ has a unique representation $v = h + k$ $\quad h \in E_1,\ k \in E_2$
we can regard v, h, k as vectors in E, or as pairs in $E_1 \oplus E_2$ as
$v = \langle h, k \rangle$, $h = \langle h, 0 \rangle$, $k = \langle 0, k \rangle$

▷ The final ingredient we need is that $D_2 f_a : E_2 \longrightarrow F$ is an iso
we know $D_2 f_a$ is linear
$D_2 f_a$ is One-to-One because $\ker(D_2 f_a) = \{0\}$.
To see this, let $v = \langle h, k \rangle$ $\quad Df_a(\langle h,k \rangle) = D_1 f_a(h) + D_2 f_a(k)$
Since $E_1 = \ker(Df_a)$ $\quad D_1 f_a(h) = 0$
Lets assume $D_2 f_a(k) = 0$ $\quad$ Then $Df_a(\langle h,k \rangle) = 0 + 0 = 0$
$\Rightarrow \langle h,k \rangle \in \ker(Df_a) \Rightarrow k = 0$

$D_2 f_a$ is Onto F because if we choose any $f \in F$
we know Df_a maps Onto F so $\exists v \ni Df_a(v) = f$
$v = \langle h, k \rangle$ so $\underbrace{Df_a(\langle h,k \rangle)}_{D_1 f_a(h) + D_2 f_a(k)} = f$, where $D_1 f_a(h) = 0$ $\quad \Rightarrow \quad D_2 f_a(k) = f$, $\quad D_2 f_a : E_2 \twoheadrightarrow F$

Therefore $D_2 f_a : E_2 \longrightarrow F$ is iso
We can apply Thm 3.2 (thickened Imp FT)



$f(a) \in W \subseteq f(\hat{\mathcal{U}} \times \hat{V}) \subseteq f(\mathcal{U})$ $\quad \square$

Now we will show that this Corollary to thickened Imp FT can be used to prove the existence of soln's to ODEs:

What we will show is that the nonlinear ODE defined on $[0,1]$ by

$$u'(t) + t\,u^2(t) = g(t) \text{ has a soln } u \in C^1[0,1] \text{ provided } |g(t)| < \varepsilon \ \forall t \in [0,1]$$

▷ Following ch 1 ex3 on sheet ② define $f: \overset{E}{C^1[0,1]} \longrightarrow \overset{F}{C^0[0,1]}$

$u \longmapsto f_u$ where $f_u(t) = u'(t) + t u^2(t)$

and we computed $Df_u(h) = h' + 2tuh$

$$\Rightarrow Df_0(h) = h' = \frac{d}{dt}h \quad \text{so} \quad Df_0 = \frac{d}{dt}$$

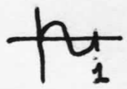
$0 \in C^1[0,1]$ (the 0 in Df_0)
$0: t \mapsto 0$

▷ Df_0 maps Onto $C^0[0,1]$: choose any $g \in C^0[0,1]$. $\exists G \ni G' = g$, namely $G(t) := \int_0^t g(t)\,dt$

▷ Split the domain:

Let $E_1 := \ker(Df_0) = \left\{ \text{all } h \in C^1 \;\middle|\; Df_0(h) = \frac{d}{dt}h = 0_{C^0} \right\} = \left\{ \text{all const fcns } h(t) = c \right\}$

$E_2 := \left\{ \text{all } h \in C^1 \;\middle|\; \int_0^1 h(t)\,dt = 0 \right\}$



Claim: $E = E_1 \oplus E_2$

pf: for any h, we can write $h = \underbrace{(h - c_h)}_{h_2} + \underbrace{c_h}_{h_1}$ where $c_h := \int_0^1 h\,dt$ const which depends on h

E_2 is a topologically clsd subsp because if $(h_k) \to h \in C^1$ we can show $\int_0^1 h\,dt = 0$

Each $h_k \in E_2$, let $h_k \to h$. Is $h \in E_2$?

This ε has no relation to ε in the thm

$k > K \Rightarrow \|h_k - h\|_{C^1} < \varepsilon$ that means $\left(\sup_{t\in[0,1]} |h_k(t) - h(t)| + \sup_{t\in[0,1]} |h_k'(t) - h'(t)| \right) < \varepsilon$

In particular, for any t, $|h_k(t) - h(t)| \le \sup_{t\in[0,1]} |h_k(t) - h(t)| < \varepsilon$

Now $\left|\int_0^1 h\,dt\right| = \left| \underbrace{\int_0^1 h_k}_{0} - \int_0^1 h \right| \le \int_0^1 |h_k - h| < \int_0^1 \varepsilon = \varepsilon$ but ε arb small

$$\Rightarrow \left|\int_0^1 h\,dt\right| = 0 \quad \square$$

Now apply Thm 3.3 taking "a" $= 0_{C^1[0,1]}$

"$f(a)$" $= f(0) = f_0$ where $f_0(t) = 0'(t) + t\,0^2(t) = 0_{\mathbb{R}}$; $f_0 = 0_{C^0}$

So by Thm 3.3 $\exists$ a ball $B(f_0, \varepsilon_1)$ in $C^0[0,1] = F \ni$ if $g \in B(0,\varepsilon)$ then $g \in f(\mathcal{U})$ ($a \in \mathcal{U}$)

[take $\mathcal{U} = E = C^1[0,1]$]

$\left\{ \text{all } g \;\middle|\; \sup_{t\in[0,1]} |g(t) - 0| < \varepsilon \right\}$

But if $g \in f(C^1[0,1])$ then $\exists\ u \in C^1[0,1] \ni f(u) = g$

i.e. $f_u = g$ i.e. $u'(t) + t u^2(t) = g(t)\ \forall t \in [0,1]$

This is saying the ODE has a soln. $\square$