Avez DC ch 3

Write-up Summary and Extensions

1/16/2024

Avez B 90

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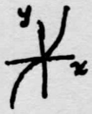
$f: \underset{\mathcal{U}}{E} \longrightarrow \underset{\mathcal{V}}{F}$ $\quad$ f is a <u>diffeo</u> if
- One-to-one
- Onto
- C^1 smooth
- f^{-1} is C^1 smooth

These are Avez's requirements

consider $f: \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^3$



He says this is a C^1 homeo, but not a diffeo because f^{-1} not difb at 0 $\quad y \longmapsto y^{1/3}$

$\triangleright$ $f^{-1} \circ f = I_E$ $\qquad$ $f \circ f^{-1} = I_F$

$D(f^{-1})_{f(x)} Df_x = I_E$ $\qquad$ $Df_{f^{-1}(y)} D(f^{-1})_y = I_F$

Rewritten: $\exists$ linear $A \ni A \cdot Df_x = I$ and $Df_x A = I$

A is left & right inv, so <u>the</u> inv.

$A = [Df_x]^{-1}$

in fact $[Df_x]^{-1} = D(f^{-1})_{f(x)}$

<u>Def</u> f is <u>slack</u> if
- f is C^1
- $Df_u : E \to F$ is an iso $\forall u \in \mathcal{U}$

$f: \mathcal{U} \to \mathcal{V}$

Summary of Contents

Detour to Cheney Applied Math

- Neumann "Geo Series" Thm: $\|A\| < 1 \Rightarrow (I-A)^{-1} = \sum^{\infty} A^k$
- Approximate Inverses: $\begin{array}{l} Ax = b \\ \|I - AB\| < 1 \end{array} \Rightarrow x = B \sum^{\infty} (I-AB)^k b$
- Invertible linear maps form an open set:
  $\begin{array}{l} A^{-1} \text{ Exists} \\ \|A-B\| < \frac{1}{\|A^{-1}\|} \end{array} \Rightarrow B^{-1}$ exists
- Linear map inversion is continuous (smooth in fact, see ch 4)
  $$\begin{aligned} J: GL(E \to F) &\longrightarrow GL(E \to F) \\ A &\longmapsto A^{-1} \quad \text{cont} \end{aligned}$$
- Contraction mapping Principle (Rudin POMA)
- Inv Fcn Thm (POMA)
- Estimate size of Nbhds in Inv FT
- 'thickened' Implicit FT
- Imp FT
- Examples, including existence of soln to ODE

Cheney AFAM p.25, 28

$T: X \to Y$ nls

$\|T\| := \sup_{\|x\| \le 1} \|Tx\|$

Thus for any $x \neq 0$

$\left\| T\left(\frac{x}{\|x\|}\right) \right\| \le \|T\| \Rightarrow \|Tx\|_Y \le \|T\|_{\mathcal{L}} \|x\|_X$ ⊛

Rename "T" as "A"

Lemma $\|A^k\| \le \|A\|^k$

Pf. observe $\|A^2 x\| = \|A(Ax)\| \underset{\text{by ⊛}}{\le} \|A\| \|Ax\| \underset{\text{⊛ again}}{\le} \|A\| \|A\| \|x\|$. Take $\sup_{\|x\| \le 1} \Rightarrow \|A^2\| \le \|A\|^2$

The rest follows inductively □

Lemma $\|A\| < 1 \Rightarrow A^m \to 0_{\mathcal{L}}$ as $m \to \infty$

Pf. Choose any $x \neq 0$ $\|A^m x\| \le \|A^m\| \|x\| \le \|A\|^m \|x\| \to 0_{\mathbb{R}}$

$\Rightarrow 0 = \lim_{m \to \infty} \|A^m x\| = \|\lim A^m x\|$ and since x is arb, we must have $\lim A^m = 0_{\mathcal{L}(X \to Y)}$ □

Thm Neumann "Geo Series" Thm

$$\left.\begin{array}{l} A \in \mathcal{L}(X \to X) \\ \|A\| < 1 \end{array}\right\} \Rightarrow \begin{array}{l} \bullet\ \exists\ (I-A)^{-1} \\ \bullet\ (I-A)^{-1} = \sum_{k=0}^{\infty} A^k \\ \bullet\ \|(I-A)^{-1}\| \le \frac{1}{1-\|A\|} \end{array}$$

This still holds for $C = -A$ $\|C\| < 1$ $\sum (-1)^k C^k$

Atkinson AITNA

For absolutely convergent series, no need to worry about pos and neg terms — Riemann Rearrangement Thm does not apply. Apostol MA p.196–197

Pf. Define partial sum $B_n := \sum_{k=0}^{n} A^k$

(B_n) is a Cauchy seq because, for $n > m$

$$\|B_n - B_m\| = \left\| \sum_{k=m+1}^{n} A^k \right\| \underset{\text{Triang}}{\le} \sum_{m+1}^{n} \|A^k\| \underset{\text{Lemma}}{\le} \sum_{m+1}^{n} \|A\|^k \le \sum_{k=m}^{\infty} \|A\|^k = \|A\|^m \sum_{k=0}^{\infty} \|A\|^k = \|A\|^m \left(\frac{1}{1-\|A\|}\right) \to 0 \text{ as } m \to \infty$$

Geo Series

$\|A\|^m \to 0$ by Lemma

Then since $\mathcal{L}$(Banach → Banach) is complete, $\exists\ B \in \mathcal{L}(X \to X) \ni (B_n) \to B$

▷ We will show $B = (I-A)^{-1}$

Right Inverse: $(I-A)B_n = B_n - AB_n = \sum_0^n A^k - A \underbrace{\sum_0^n A^k}_{\sum_{k=1}^{n+1} A^k} = I - A^{n+1}$

Then $\lim_{n \to \infty} (I-A) B_n = \lim_{n \to \infty} I - \cancel{A^{n+1}}^{\,0}$

$\Rightarrow (I-A)B = I$

Likewise for Left Inv: $\lim B_n (I-A) = \lim I - A^{n+1} \Rightarrow B(I-A) = I$

$\Rightarrow B = (I-A)^{-1}$ □

To show the bounds: $\|B\| = \left\| \sum_0^\infty A^k \right\| \le \sum_0^\infty \|A\|^k = \frac{1}{1-\|A\|}$ by Geo Series □

▷ Let's use this to solve $(I-A)x = v$

example 1 solve for fcn x: $x(t) - \lambda \underbrace{\int_0^1 K(s,t)\, x(s)\, ds}_{(A(x))(t)} = v(t)$

Functional eq: $(I - \lambda A)(x) = v$ if we are in a suitable Banach sp and if $\|\lambda A\| < 1$ we can compute $[I - \lambda A]^{-1}$ by Neumann

$$x = [I - \lambda A]^{-1} v = \sum_{n=0}^{\infty} (\lambda A)^n v = v + \lambda A v + \lambda^2 A^2 v + \ldots$$

Make this more concrete in example 2 →

Cheney p.187

10/24/2023 ②

ex 2 A more specific version of ex 1 : $x(t) - \lambda \int_0^1 e^{t-s} x(s)\,ds = v(t)$

$(Ax)(t) := \int_0^1 e^{t-s} x(s)\,ds$

Show $A^2 = A$:

$$(A^2x)(t) = \int_{s=0}^1 e^{t-s}(Ax)(s)\,ds = \int_{s=0}^1 e^{t-s}\int_{\sigma=0}^1 e^{s-\sigma} x(\sigma)\,d\sigma\,ds \overset{\text{interchange}}{=} \int_{\sigma=0}^1\int_{s=0}^1 e^{t-s+s-\sigma} x(\sigma)\,ds\,d\sigma$$

$$= \int_{\sigma=0}^1 e^{t-\sigma} x(\sigma)\left[\int_0^1 ds\right] d\sigma = \int_{\sigma=0}^1 e^{t-\sigma} x(\sigma)\,d\sigma = (Ax)(t)$$

Thus $x = [I-\lambda A]^{-1} v = \sum_{n=0}^{\infty} (\lambda A)^n v = Iv + \left(\sum_{n=1}^{\infty} \lambda^n\right) Av = v + \left(\frac{1}{1-\lambda} - 1\right) Av = v + \frac{\lambda}{1-\lambda} Av$ □

▷ Approximate Inverses

We want to solve $Ax = b$ and we have a linear map $B \ni$ "$AB \approx I$"
Then B is an 'Approx Right Inv' to A and we can use it to solve for x iteratively.

Thm $Ax = b$, $\exists B \ni \|I - AB\| < 1$ $\Rightarrow$

- $B \approx A^{-1}$ Approx Right Inv
- $x = B(AB)^{-1} b = B \sum_{k=0}^{\infty} (I-AB)^k b$
- We can solve for x recursively: $x_0 := Bb$, $x_{n+1} := x_n + B(\underbrace{b - Ax_n}_{\text{residual}})$

This is a big subject in Atkinson AITNA circa p.425

Pf. By Neumann's "Geo Series" Thm

$\|\mathcal{A}\| < 1 \Rightarrow \exists [I-\mathcal{A}]^{-1}$ and $[I-\mathcal{A}]^{-1} = \sum_0^{\infty} \mathcal{A}^k$

So here $\mathcal{A} = I - AB$

$[I-\mathcal{A}]^{-1} = \sum \mathcal{A}^k \Rightarrow [I-(I-AB)]^{-1} = \sum_{k=0}^{\infty} (I-AB)^k \Rightarrow (AB)^{-1} = \sum_{k=0}^{\infty} (I-AB)^k$

We can see $x := B(AB)^{-1} b$ is a soln because $Ax = AB(AB)^{-1} b = Ib = b$

Thus $x = B(AB)^{-1} b = B\left[\sum_{k=0}^{\infty} (I-AB)^k\right] b = B\left[I + (I-AB) + (I-AB)^2 + \cdots\right] b$
$= Bb + B(I-AB)b + B(I-AB)^2 b + \cdots$

▷ We want the recursion: $\begin{cases} x_0 := Bb \\ x_{n+1} := x_n + B(b - Ax_n) \end{cases}$

Define partial sums $y_n := B \sum_{k=0}^{n} (I-AB)^k b$

We want to show $x_n = y_n$ for all n and we will do this by induction:

To start $x_0 = Bb$

$$x_1 = x_0 + B(b - Ax_0) = Bb + B(b - ABb) = B[\underbrace{I + (I-AB)}_{\sum_{k=0}^{1}(I-AB)^k}]b$$

Now for induction step ⟶

Induction step:

Assume $x_n = y_n$ and show $x_{n+1} = y_{n+1}$

Let $S_n := \sum_{0}^{n} (I-AB)^k$ so $y_n = BS_n b$

$$x_{n+1} = x_n + B(b - Ax_n)$$

hypoth $\to$ $x_n \to y_n \to BS_n b$, and $B(b - Ax_n) \to Bb - BAy_n$

$$BS_n b + Bb - BA(BS_n b) = B\left[S_n + I - ABS_n\right] b$$
$$= B\left[(I-AB)S_n + I\right] b$$
$$= B\Big[\underbrace{(I-AB)\sum_{k=0}^{n}(I-AB)^k + I}_{\sum_{k=0}^{n}(I-AB)^{k+1} + I}\Big] b$$
$$\underbrace{\sum_{k=1}^{n+1}(I-AB)^k + I}_{\sum_{k=0}^{n+1}(I-AB)^k}$$
$$= BS_{n+1} b$$
$$= y_{n+1} \checkmark \quad \square$$

$\triangleright$ Now we get to the point of all this:

A is an invertible map $\Rightarrow$ all linear maps near A are also invertible.
$\Rightarrow$ A is an interior pt of $GL(X \to Y)$
$\Rightarrow$ $GL(X \to Y)$ is open in $\mathcal{L}(X \to Y)$

cheney AFAM p.189

Thm (Invertible Linear Maps Form an Open Set)

$$\left.\begin{array}{l} A^{-1} \text{ exists} \\ \|A-B\| < \frac{1}{\|A^{-1}\|} \end{array}\right\} \Rightarrow B^{-1} \text{ exists}$$

pf. $B = A + B - A$
$$= A\left[I + A^{-1}(B-A)\right] = A\left[I - \underbrace{A^{-1}(A-B)}\right]$$

show this is $\|\cdot\| < 1$ and apply Neumann 'Geo Series' Thm

$\|A^{-1}(A-B)\| \le \|A^{-1}\|\|A-B\| < 1$ by hypoth

Then take $\mathcal{A} := A^{-1}(A-B)$

Neumann $\Rightarrow$ $[I-\mathcal{A}]^{-1}$ exists i.e. $\left[I - A^{-1}(A-B)\right]^{-1}$ exists

Thus $B = A\left[I - A^{-1}(A-B)\right]$

is the composition of 2 invertible maps, thus B^{-1} exists

$$B^{-1} = \left[I - A^{-1}(A-B)\right]^{-1} A^{-1} \quad \blacksquare$$

Atkinson AITNA p. 425 also gives

$$\|B^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|A-B\|} \quad \text{and} \quad \|A^{-1} - B^{-1}\| \le \frac{\|A^{-1}\|^2 \|A-B\|}{1 - \|A^{-1}\|\|A-B\|}$$

Now, following Rudin POMA p.209

Thm (Liner Map Inversion is Cont) The map $J: GL(X\to Y) \longrightarrow GL(X\to Y)$ is Cont
$$A \longmapsto A^{-1}$$

pf. A^{-1} exists, so we define $\alpha := \frac{1}{\|A^{-1}\|}$

We must show: Given $\varepsilon > 0$, $\exists\, \delta > 0 \ni \|A-B\| < \delta \Rightarrow \|J(B) - J(A)\| < \varepsilon$
i.e. $\|B^{-1} - A^{-1}\| < \varepsilon$

we also take $\delta < \alpha$

Observe $B^{-1} - A^{-1} = B^{-1}(A-B)A^{-1}$

(★) $\|B^{-1} - A^{-1}\| \le \|B^{-1}\| \|A-B\| \|A^{-1}\|$ we know $\|A-B\| < \delta$ and $\|A^{-1}\| = \frac{1}{\alpha}$ so we only need to bound $\|B^{-1}\|$.

To bound B^{-1}, we need to develop an ineq: For any $x \in X$

$\alpha\|x\| = \alpha|A^{-1}Ax| \le \alpha|A^{-1}||Ax| \le \alpha\frac{1}{\alpha}|Ax| = |(A-B+B)x| \le |(A-B)x| + |Bx|$
$\le \delta|x| + |Bx|$

$\Rightarrow (\alpha - \delta)\|x\| \le \|Bx\|$

Now since B^{-1} exists, $\exists\, y \ni x = B^{-1}y$. Plug this in:

$(\alpha-\delta)\|B^{-1}y\| \le \|BB^{-1}y\| = \|y\|$

$(\alpha-\delta)\sup\limits_{|y|=1}\|B^{-1}y\| \le \sup\limits_{|y|=1}\|y\| \Rightarrow (\alpha-\delta)\|B^{-1}\| \le 1 \Rightarrow \|B^{-1}\| \le \frac{1}{\alpha-\delta}$

Plugging into (★): $\|B^{-1} - A^{-1}\| \le \|B^{-1}\|\|A-B\|\|A^{-1}\| \le \frac{1}{(\alpha-\delta)}\frac{\delta}{\alpha} =: \varepsilon$

we want $\varepsilon > 0$ given, and δ to be determined by that: $\varepsilon = \frac{\delta}{(\alpha-\delta)\alpha} \Rightarrow \delta = \frac{\varepsilon\alpha^2}{(1+\varepsilon\alpha)}$

so take this δ and the result follows. □

Since I went to the trouble of decrypting Avez Lemma 3.2, I will also include that here. The notation is different, but it is essentially the same pfs as the last few sheets!

Avez p. 25 **Lemma 3.2** $GL(E\to F)$ is open in $\mathcal{L}(E\to F)$

$J: GL(E\to F) \longrightarrow GL(E\to F)$ is Cont
$$u \longmapsto u^{-1}$$

$$\begin{tikzcd} & F \arrow[dr, "v^{-1}"] & \\ E \arrow[ur, "g"] \arrow[rr, "u"] & & E \end{tikzcd}$$

Fixed, invertible map

pf. We can assume $E = F$ and thus we are working with $GL(E\to E)$ because

Say we want to show g^{-1} exists. Define $u := v^{-1}\circ g$ then $g = v\circ u$ and $g^{-1} = u^{-1}\circ v^{-1}$

Thus g^{-1} exists $\Leftrightarrow$ u^{-1} exists since v is a known invertible, and specific map.

So assume $u \in GL(E\to E)$ and $h \in \mathcal{L}(E\to E)$. We shall prove $\|h\| < \frac{1}{\|u^{-1}\|} \Rightarrow u+h \in GL(E\to E)$

$u+h = u[I + u^{-1}h]$ (we know u^{-1} exists)

Then $(u+h)^{-1} \overset{!}{=} [I+u^{-1}h]^{-1}u^{-1}$ Therefore it is enough to show $[I+u^{-1}h]^{-1}$ exists

Define $v := -u^{-1}h$ <u>CLAIM</u>: $[I-v]^{-1}$ exists if $\|v\| < 1$ observe $\|v\| = |-u^{-1}h| \le |u^{-1}||h| < 1$

Let $X_n := \sum\limits_{k=0}^{n} v^k$

This is Cauchy seq because $\|X_{p+q} - X_p\| = |v^{p+q} + \dots + v^{p+1}\| \le \sum\limits_{k=p+1}^{p+q}|v|^k \longrightarrow 0$ as $p\to\infty$

$\mathcal{L}(E\to E)$ is Complete $\Rightarrow X := \lim X_n$ exists as elt of $\mathcal{L}(E\to E)$

$(I-v)X_n = I - v^{n+1}$ Take lim: $(I-v)X = I$

This establishes $GL(E\to E)$ is open

Same arg as in Neumann Geo Series pf. earlier

Cont'd ⟶

Now show J is Cont:

$$J(u+h) - J(u) = (u+h)^{-1} - u^{-1}$$
$$= [I-v]^{-1}u^{-1} - u^{-1} = [(I-v)^{-1} - I]u^{-1} = [\lim_n X_n - I]u^{-1}$$
$$= \left[\lim_n \sum_1^n v^k\right]u^{-1}$$

Thus
$$\|J(u+h) - J(u)\| \le \left(\lim \sum^n \|v^k\|\right)|u^{-1}| \le \left(\lim \sum^n \|v\|^k\right)|u^{-1}| \le \frac{\|v\|}{1-\|v\|}\|u^{-1}\|$$
$$\le \frac{\|h\|\,\|u^{-1}\|^2}{1-\|u^{-1}h\|}$$

Continuity follows if $\lim_{\|h\|\to 0} \|J(u+h) - J(u)\| = 0$

and this is true because $\frac{\|h\|\,\|u^{-1}\|^2}{1-\|u^{-1}h\|} \to \frac{0}{1} = 0$ $\quad u^{-1}h \to 0$ because $u^{-1}(\cdot)$ is continuous and linear □

△ Jumping out of seq with Avez, we will now show J is C^∞ smooth.
This is from Ch 4, and assumes we know about ~~2nd deriv~~

$$\boxed{\text{p.39 Thm 4.5} \quad \left.\begin{array}{l} J: GL_{EF} \longrightarrow GL_{EF} \\ \quad u \longmapsto u^{-1} \end{array}\right\} \Rightarrow \begin{array}{l} J \text{ is } C^\infty \text{ smooth} \\ DJ_u(h) = -u^{-1}\circ h \circ u^{-1} \end{array} \quad \text{See ch 4 sheets 8 - 10}}$$

Pf.

Step 1 what is $DJ_u(h)$? Guess: $DJ_u(h) =: L(h) := -u^{-1}\circ h\circ u^{-1}$ (not conj.!)

Check: $J(u+h) - J(u) - L(h) = (u+h)^{-1} - u^{-1} + u^{-1}hu^{-1}$
$$= (u+h)^{-1}\left[I - (u+h)u^{-1} + (u+h)u^{-1}hu^{-1}\right]$$
$$\cancel{I} - \cancel{uu^{-1}} - \cancel{hu^{-1}} + \underbrace{uu^{-1}}_{I}\cancel{hu^{-1}} + hu^{-1}hu^{-1} = (u+h)^{-1}[hu^{-1}hu^{-1}]$$

then
$$\frac{\|J(u+h) - Ju - Lh\|}{\|h\|} = \frac{|(u+h)^{-1}[hu^{-1}hu^{-1}]|}{|h|} \le \frac{|(u+h)^{-1}|\,\cancel{|h|}\,|u^{-1}|\,|h|\,|u^{-1}|}{\cancel{|h|}} = \underbrace{|(u+h)^{-1}|}_{\substack{\text{This goes} \\ \text{to } u^{-1} \text{ since} \\ J \text{ is Cont, by prev result.}}}|u^{-1}|^2|h| \to 0 \text{ as } |h| \to 0$$

Step 2 To start the induction, we will show J is C^1

Define $f_{ab}: \mathcal{L}(E\to F) \longrightarrow \mathcal{L}(E\to F)$ for $a, b \in \mathcal{L}(E\to F)$
$\qquad h \longmapsto -a\circ h\circ b$ $\quad$ This would allow us to write $DJ_u(\cdot) = f_{u^{-1}u^{-1}}(\cdot)$

Define $f: \mathcal{L}(E\to F)\times\mathcal{L}(E\to F) \longrightarrow \mathcal{L}(\mathcal{L}_{EF}\to\mathcal{L}_{EF})$
$\qquad a, b \longmapsto f_{ab}(\cdot)$ $\quad$ This is bilinear and Cont because Bdd Linear:
$\|f(a,b)\| = \|f_{ab}\|_{op} = \sup_{|h|=1} |ahb| \le \sup_{h=1} |a||h||b| = |a||b|$

Define $\mathcal{J}: GL(E\to F) \longrightarrow \mathcal{L}(E\to F)\times\mathcal{L}(E\to F)$
$\qquad u \longmapsto (u^{-1}, u^{-1}) = (Ju, Ju)$ so this is Cont and as smooth as J is.

Then $DJ = f\circ\mathcal{J}: u \xmapsto{\mathcal{J}} (u^{-1}, u^{-1}) \xmapsto{f} f_{u^{-1}u^{-1}}(\cdot)$ so it is linear and cont — composition of linear, Cont maps.
$\Rightarrow J$ is C^1

Step 3

$$\boxed{\begin{array}{l}\text{Stopping this discussion here} \\ \text{It is done in ch 4 sheets 8 - 10}\end{array}}$$

Rudin POMA ch 9
p. 220-221

10/13/2023

Remarks
- A contraction squeezes any 2 pts closer together
- Iterating the contraction infinitely many times squashes X to a pt x^*, the FP

5

Contraction Mapping Principle

X metric sp w/ metric d

Contraction cond

Def $\varphi: X \to X$ is a Contraction if $\exists$ Real $\boxed{c<1} \ni d(\varphi(x), \varphi(y)) \leq c\, d(x,y) \;\forall x,y \in X$

$$\boxed{\text{FP Thm}: \left.\begin{array}{l} X \text{ complete metric sp.} \\ \varphi: X \to X \text{ contraction} \end{array}\right\} \Rightarrow \begin{array}{l} \exists \text{ unique } x^* \in X \ni \varphi(x^*) = x^* \quad \text{F.P.} \\ \varphi \text{ is unif cont on } X \end{array}}$$

Pf. choose arb $x_0 \in X$ and define a seq (x_n) by $x_1 = \varphi(x_0)$, $x_{n+1} = \varphi(x_n)$
We shall show (x_n) is a Cauch seq, and thus, since X is complete, $(x_n) \to x^* \in X$.

Observe $d(\overset{\varphi(x_n)}{x_{n+1}}, \overset{\varphi(x_{n-1})}{x_n}) = d(\varphi(x_n), \varphi(x_{n-1})) \leq c\, d(x_n, x_{n-1})$ by contraction cond

then (★) $d(x_{n+1}, x_n) \leq c\, d(x_n, x_{n-1}) \leq \dots \leq c^n d(x_1, x_0)$

For Cauch seq, let large pos integers $n < m$

$$d(x_m, x_n) \underset{\text{Triang ineq}}{\leq} \underbrace{d(x_{n+1}, x_n)}_{\leq c^n d(x_1,x_0)} + \underbrace{d(x_{n+2}, x_{n+1})}_{c^{n+1} d(x_1,x_0)} + \dots + \underbrace{d(x_m, x_{m-1})}_{c^{m-1} d(x_1,x_0)} \text{ by (★)}$$

$$\leq \underbrace{(c^n + c^{n+1} + \dots + c^{m-1})}_{(1 + c + \dots + c^{(m-1)-n}) c^n} d(x_1, x_0)$$

$p = m-n-1$

Finite Geo sums:
$S = 1 + a + a^2 + \dots + a^p$
$\Rightarrow aS = a + a^2 + \dots + a^{p+1}$
Subtract $S - aS = 1 - a^{p+1}$
$S = \frac{1 - a^{p+1}}{1-a}$

Finite geo sum $\left[\frac{1 - c^{p+1}}{1-c}\right] c^n = \left[\frac{1 - c^{m-n}}{1-c}\right] c^n$

$$\Rightarrow d(x_m, x_n) \leq \left[\frac{1 - c^{m-n}}{1-c}\right] c^n d(x_1, x_0) < \left[\frac{1}{1-c}\right] d(x_1, x_0) c^n$$

Since $c < 1$, c^n arb small for n large enough
$\Rightarrow (x_n)$ is Cauchy seq.

Since X complete, $\exists\, x^* \in X$ such than $(x_n) \to x^*$

▷ Is x^* unique? Yes.

Suppose for some initial pts x_0, y_0 $(\varphi^n(x_0)) \to x^*$ while $(\varphi^n(y_0)) \to y^*$ $\quad \varphi(x^*) = x^*$, $\varphi(y^*) = y^*$

By the contraction cond $d(\underset{x^*}{\varphi(x^*)}, \underset{y^*}{\varphi(y^*)}) \leq c\, d(x^*, x^*)$

$\Rightarrow d(x^*, y^*) \leq c\, d(x^*, y^*)$ only possible if $d(x^*, y^*) = 0$
$\Rightarrow x^* = y^*$.

▷ φ is Unif Cont in X

Given $\varepsilon > 0$, $\exists\, \delta > 0$ (indep of x,y) $\ni$ for any $x,y \in X$
$d(\varphi(x), \varphi(y)) < \varepsilon$ if $d(x,y) < \delta$

By contraction cond $d(\varphi(x), \varphi(y)) \leq c\, d(x,y) < c\delta$ so take $\delta = \varepsilon/c$

▷ $\varphi(x^*) = \varphi(\lim x_n) = \lim \varphi(x_n) = \lim x_{n+1} = x^*$

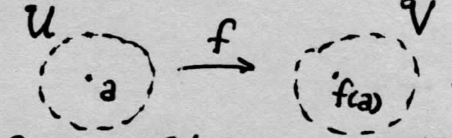
$\boxed{\text{QED}}$

Rudin POMA ch 9
P. 221-223

10/17/2023

⑥

Inv Fcn Thm



Thm X, Y Banach
$f: \overset{X}{\mathcal{O}} \longrightarrow Y$ C^N smooth
$A^{-1} := [Df_a]^{-1}$ exists for one $a \in \mathcal{O}$
(In fin dim case, this forces dim X = dim Y

$\Rightarrow$ $f: \mathcal{U} \to \mathcal{V}$ is a C^N diffeo
In more detail: • f^{-1} exists near $f(a)$ (i.e. in $\mathcal{V}$)
• $D(f^{-1})_{f(x)}$ exists in $\mathcal{V}$
• $D(f^{-1})_{f(x)} = [Df_x]^{-1}$ $x \in \mathcal{U}$
• f^{-1} is C^N if f C^N

Df_a dominates the behaviour of f near a

Pf. Step 1 Establish f is One-to-One in a nbhd of a [i.e. f^{-1} exists there]

Since Df_x is a cont fcn of x, Given $\lambda > 0$ $\exists\, \delta > 0 \ni$
$x \in \overset{\circ}{B}(a,\delta) \Rightarrow |Df_x - Df_a| < \lambda$ Take $\lambda := \frac{1}{2\|\cancel{Df_a}\|}$ A^{-1} i.e. $\|\cancel{Df_a}\| = \frac{1}{2\lambda}$ A^{-1}
Then for the associated δ $\mathcal{U} := \overset{\circ}{B}(a,\delta)$

A = fixed map Df_a
$A^{-1} = [Df_a]^{-1}$

▷ For each $y \in Y$, define maps $\varphi_y : \mathcal{U} \longrightarrow X$
$x \longmapsto x + Df_a^{-1}(y - f(x))$

(*) observe $f(x) = y \Leftrightarrow \varphi_y(x) = x$ F.P.

Fix y
▷ Show φ_y satisfies Contraction cond: $\|\varphi_y(x_1) - \varphi_y(x_2)\| \le \frac{1}{2}\|x_1 - x_2\|$

From MVT (Avez DC ch 1 sheet ⑥C) : $\|\varphi_y(x_2) - \varphi_y(x_1)\| \le \|D(\varphi_y)_x\|_{op}\|x_2 - x_1\|$
$D(\varphi_y)_x = I - Df_a^{-1}Df_x = Df_a^{-1}[Df_a - Df_x]$
then $\|D(\varphi_y)_x\| \le \|Df_a^{-1}\| \cdot \|Df_a - Df_x\| < \frac{1}{2\lambda}\lambda = \frac{1}{2}$ since $x \in \mathcal{U} = \overset{\circ}{B}(a,\delta)$

▷ φ_y can have at most one FP in $\mathcal{U}$ because if $\varphi(x_1) = x_1$ and $\varphi(x_2) = x_2$ $x_1 \ne x_2$
$\|\underset{x_1}{\varphi(x_1)} - \underset{x_2}{\varphi(x_2)}\| \le \frac{1}{2}\|x_1 - x_2\| \Rightarrow \|x_1 - x_2\| \le \frac{1}{2}\|x_1 - x_2\|$ This is only possible if $\|x_1 - x_2\| = 0$
i.e. $x_1 = x_2$.

We have now show f is One-to-One in $\mathcal{U}$ because if $f(x_1) = y$ and $f(x_2) = y$ by (*)
then $\varphi(x_1) = x_1$ and $\varphi(x_2) = x_2$ which we have just shown forces $x_1 = x_2$

Step 2 Define $\mathcal{V} = f(\mathcal{U})$. Show $\mathcal{V}$ is open in Y by showing every pt is an interior pt.

choose any $y_0 \in f(\mathcal{U})$. Then $\exists\, x_0 \ni f(x_0) = y_0$
Since $\mathcal{U}$ is open, $\exists\, r > 0 \ni \bar{B}(x_0, r) \subset \mathcal{U}$.

We will show y_0 is an interior pt of $\mathcal{V}$ by showing every pt y in $B(y_0, \lambda r)$
is also the image of some x: $f(x) = y$.

Let y be any pt in $B(y_0, \lambda r)$

▷ observe $\varphi_y : \overline{B}(x_0, r) \longrightarrow \overline{B}(x_0, r)$ because, for any $x \in \overline{B}$, $\|\varphi_y(x) - x_0\| < r$:

$$\varphi_y(x) - x_0 = \varphi_y(x) - \varphi_y(x_0) + \varphi_y(x_0) - x_0$$

$$\|\varphi_y(x) - x_0\| \le \underbrace{\|\varphi_y(x) - \varphi_y(x_0)\|}_{\substack{\le \frac{1}{2}\|x - x_0\| \\ < \frac{1}{2} r}} + \underbrace{\|\varphi_y(x_0) - x_0\|}$$

$$= \|\cancel{x_0} - Df_a^{-1}(y - f(x_0)) - \cancel{x_0}\|$$
$$= \|Df_a^{-1}(y - y_0)\|$$
$$\le \|Df_a^{-1}\|_{op} \, \|y - y_0\|$$
$$< \frac{1}{2\lambda} \quad \lambda r \quad = \frac{1}{2} r$$

so $\varphi_y(x) \in \overline{B}$

▷ Since $\overline{B}$ is a complete metric sp, we can apply Contraction Map FP Thm
$\Rightarrow \exists! \; x_y^* \in \overline{B} \ni \varphi_y(x_y^*) = x_y^*$ and by ⓐ $f(x_y^*) = y$ so $B(y_0, \lambda r)$ contains only pts of $f(\mathcal{U})$

<u>Step 3</u> we know $f : \overset{open}{\mathcal{U}} \to \overset{open}{\mathcal{V}}$ and f^{-1} exists there. Now we must show f^{-1} difb.

▷ Show $[Df_x]^{-1}$ exists and is bdd.
we have $\|Df_x - Df_a\| < \lambda$ where $\lambda = \frac{1}{2\|A^{-1}\|}$ $A := Df_a$
Thus $\|Df_x - Df_a\| < \lambda < 2\lambda = \frac{1}{\|A^{-1}\|}$ i.e. $\|Df_x - Df_a\| \, \|A^{-1}\| < 1$ see sheet ____
we can apply <u>Thm Invertible Maps form an Open Set</u> and get:
linear map $T := [Df_x]^{-1}$ exists and $\|T\| \le \dfrac{\|A^{-1}\|}{1 - \|A^{-1}\| \, \|Df_x - Df_a\|}$

▷ But is $[Df_x]^{-1} = D(f^{-1})_{f(x)}$? Do we even know RHS exists?
Lets denote $g := f^{-1}$. The defining cond for Dg_y to exist is $\exists$ linear map $L \ni$

$$\lim_{k \to 0} \frac{\|g(y+k) - g(y) - Lk\|}{\|k\|} = 0$$

we will plug in T for L and thus show $L = Dg_y$ exists and $Dg_y = T$

Choose $y \in \mathcal{V}$ and k small enough such that $y + k \in \mathcal{V}$
Then $\exists \; x, x+h$ where $f(x) = y$ and $f(x+h) = y+k$ NOTE: $f(x+h) - f(x) = y + k - y = k$

$$g(y+k) - g(y) - Tk = x + h - x - Tk = h - Tk$$
$$= TT^{-1}h - Tk = T[\underset{Df_x h}{T^{-1}h} - \underset{(f(x+h) - f(x))}{k}]$$
$$= -T[f(x+h) - f(x) - Df_x h]$$

cont'd →

Take norms and divide by $\|k\|$:

$$\frac{\|g(y+k)-g(y)-Tk\|}{\|k\|} \leq \frac{\|T\|\,\|f(x+h)-f(x)-Df_x h\|}{\|k\|} \underset{\uparrow}{\leq} \frac{\|T\|}{\lambda}\left[\frac{\|f(x+h)-f(x)-Df_x h\|}{\|h\|}\right]$$

CLAIM: $\|h\| \leq \frac{1}{\lambda}\|k\|$

Once we establish the claim,
Then $|k| \to 0 \Rightarrow |h| \to 0$ Then the RHS of above $\underset{0}{\text{ineq}} \to 0$ and thus the LHS also.
Thus g is dif'b (hence Cont) and $Dg_y = T$ i.e. $D(f^{-1})_{f(x)} = [Df_x]^{-1}$

Δ Establish Claim: $\|h\| \leq \frac{1}{\lambda}\|k\|$

For any $z \in \mathcal{V}$
$$\varphi_z(x+h) = x+h + A^{-1}[z - f(x+h)]$$
$$\varphi_z(x) = x + A^{-1}[z - f(x)]$$
$$\varphi_z(x+h) - \varphi_z(x) = h + A^{-1}[\underbrace{-f(x+h)}_{-(y+k)} + \underbrace{f(x)}_{y}] = h - A^{-1}k$$

Thus $\|\varphi_z(x+h) - \varphi_z(x)\| = \|h - A^{-1}k\|$

We also have the contraction Cond: $\|\varphi_z(x+h) - \varphi_z(x)\| \leq \frac{1}{2}\|x+h-x\| = \frac{1}{2}\|h\|$

$\Rightarrow \|h - A^{-1}k\| \leq \frac{1}{2}\|h\|$

Reverse Triang: $\|h\| - \|A^{-1}k\| \leq \frac{1}{2}\|h\| \Rightarrow -\|A^{-1}k\| \leq -\frac{1}{2}\|h\| \Rightarrow \|h\| \leq 2\|A^{-1}k\| \leq 2\|A^{-1}\|\,\|k\|$
$$= \cancel{2}\left(\frac{1}{\cancel{2}\lambda}\right)\|k\|$$
$$= \frac{1}{\lambda}\|k\|$$

$\boxed{\text{END CLAIM}}$

The last result is $f \in C^N \Rightarrow f^{-1} \in C^N$

We have $D(f^{-1})_{f(x)} = [Df_x]^{-1}$ or for $y = f(x)$ $D(f^{-1})_y = [Df_{f^{-1}(y)}]^{-1}$
$$= J \circ Df \circ f^{-1}$$

We showed J is C^∞ ~~earlier in these sheets, and~~ in ch 4. (no, only ch 4)

Up to now we have only established f is difb in $\mathcal{U}$, Thus $D(f^{-1})_y = \underset{C^\infty}{J} \circ \underset{\text{exists}}{Df} \circ \underset{\text{difb}}{f^{-1}}$

So we just can repeat $D(f^{-1})_y$ exists in $\mathcal{V}$.

If f would be C^1, then $D(f^{-1}) = \underset{C^\infty}{J} \cdot \underset{C^0}{Df} \cdot \underset{\text{difb}}{f^{-1}} \Rightarrow D(f^{-1})$ is $C^0 \Rightarrow f^{-1}$ is C^1

Assume $f\ C^{N-1} \Rightarrow f^{-1}\ C^{N-1}$

By the same arg, if f is C^N, $D(f^{-1}) = J \cdot \underset{C^{N-1}}{Df} \cdot \underset{C^{N-1}}{f^{-1}} \Rightarrow D(f^{-1})\ C^{N-1} \Rightarrow f^{-1}\ C^N$

$\boxed{\text{QED}}$