10/29/2023

Now, following Rudin POMA p.209

$\boxed{\underline{\text{Thm}} \text{ (Liner Map Inversion is Cont)} \quad \text{The map } J: GL(X\to Y) \longrightarrow GL(X \to Y) \text{ is Cont}, \quad A \longmapsto A^{-1}}$

pf. A^{-1} exists, so we define $\alpha := \frac{1}{\|A^{-1}\|}$

we must show: Given $\varepsilon > 0$, $\exists\, \delta > 0 \ni \|A-B\| < \delta \Rightarrow \|J(B) - J(A)\| < \varepsilon$

i.e $\|B^{-1} - A^{-1}\| < \varepsilon$

Observe $B^{-1} - A^{-1} = B^{-1}(A-B)A^{-1}$ we also take $\delta < \alpha$

(★) $\|B^{-1} - A^{-1}\| \leq \|B^{-1}\| \|A-B\| \|A^{-1}\|$ we know $\|A-B\| < \delta$ and $\|A^{-1}\| = \frac{1}{\alpha}$ so we only need to bound $\|B^{-1}\|$.

To bound B^{-1}, we need to develop an ineq: For any $x \in X$

$\alpha\|x\| = \alpha|A^{-1}Ax| \leq \alpha|A^{-1}||Ax| \leq \alpha\frac{1}{\alpha}|Ax| = |(A-B+B)x| \leq |(A-B)x| + |Bx|$

$\leq \delta|x| + |Bx|$

$\Rightarrow (\alpha - \delta)\|x\| \leq \|Bx\|$

Now since B^{-1} exists, $\exists\, y \ni x = B^{-1}y$. Plug this in:

$(\alpha-\delta)\|B^{-1}y\| \leq \|BB^{-1}y\| = \|y\|$

$(\alpha - \delta)\sup_{|y|=1}\|B^{-1}y\| \leq \sup_{|y|=1}\|y\| \Rightarrow (\alpha-\delta)\|B^{-1}\| \leq 1 \Rightarrow \|B^{-1}\| \leq \frac{1}{\alpha - \delta}$

Plugging into (★): $\|B^{-1} - A^{-1}\| \leq \|B^{-1}\| \|A-B\| \|A^{-1}\| \leq \frac{1}{(\alpha-\delta)}\frac{\delta}{\alpha} =: \varepsilon$

we want $\varepsilon > 0$ given, and δ to be determined by that: $\varepsilon = \frac{\delta}{(\alpha-\delta)\alpha} \Rightarrow \delta = \frac{\varepsilon\alpha^2}{(1+\varepsilon\alpha)}$

so take this δ and the result follows. □

Since I went to the trouble of decrypting Avez Lemma 3.2, I will also include that here.
The notation is different, but it is essentially the same pfs as the last few sheets!

Avez p. 25 $\underline{\text{Lemma 3.2}}$ $GL(E\to F)$ is open in $\mathcal{L}(E\to F)$

$J: GL(E\to F) \longrightarrow GL(E\to F)$ is Cont

$u \longmapsto u^{-1}$

Fixed, invertible map

$$\begin{tikzcd} & F \arrow[dr, "v^{-1}"] & \\ E \arrow[ur, "q"] \arrow[rr, "u"] & & E \end{tikzcd}$$

pf. We can assume $E = F$ and thus we are working with $GL(E\to E)$ because
Say we want to show q^{-1} exists. Define $u := v^{-1} \circ q$ then $q = v \circ u$ and $q^{-1} = u^{-1} \circ v^{-1}$
Thus q^{-1} exists $\Leftrightarrow$ u^{-1} exists since v is a known invertible, and specific map.

So assume $u \in GL(E\to E)$ and $h \in \mathcal{L}(E\to E)$. We shall prove $\|h\| < \frac{1}{\|u^{-1}\|} \Rightarrow u+h \in GL(E\to E)$

$u+h = u[I + u^{-1}h]$ (we know u^{-1} exists)

Then $(u+h)^{-1} \stackrel{!}{=} [I+u^{-1}h]^{-1}u^{-1}$ Therefore it is enough to show $[I+u^{-1}h]^{-1}$ exists

Define $v := -u^{-1}h$ $\underline{\text{CLAIM}}$: $[I-v]^{-1}$ exists if $\|v\| < 1$ observe $\|v\| = |-u^{-1}h| \leq |u^{-1}||h| < 1$

Let $X_n := \sum_{k=0}^{n} v^k$

This is Cauchy seq because $\|X_{p+q} - X_p\| = \|v^{p+q} + \dots + v^{p+1}\| \leq \sum_{k=p+1}^{p+q} |v|^k \longrightarrow 0$ as $p \to \infty$

$\mathcal{L}(E\to E)$ is Complete $\Rightarrow$ $X := \lim X_n$ exists as elt of $\mathcal{L}(E\to E)$

$(I - v)X_n = I - v^{n+1}$ Take lim: $(I-v)X = I$

This establishes $GL(E\to E)$ is open

Same arg as in Neumann Geo Series pf. earlier

Cont'd ⟶

Rudin POMA ch 9
p. 220-221

10/13/2023

Remarks
- A contraction squeezes any 2 pts closer together
- Iterating the contraction infinitely many times squashes X to a pt x^*, the FP

Contraction Mapping Principle

X metric sp w/ metric d

Def $\varphi : X \to X$ is a Contraction if $\exists$ Real $\boxed{c<1}$ $\ni$ $\underbrace{d(\varphi(x), \varphi(y)) \le c\, d(x,y)}^{\text{Contraction cond}}$ $\forall x,y \in X$

FP Thm: $\left.\begin{array}{l} X \text{ complete metric sp.} \\ \varphi : X \to X \text{ contraction} \end{array}\right\} \Rightarrow \begin{array}{l} \exists \text{ unique } x^* \in X \ni \varphi(x^*) = x^* \quad \text{F.P.} \\ \varphi \text{ is unif cont on } X \end{array}$

Pf. choose arb $x_0 \in X$ and define a seq (x_n) by $x_1 = \varphi(x_0)$, $x_{n+1} = \varphi(x_n)$

We shall show (x_n) is a Cauch seq, and thus, since X is complete, $(x_n) \to x^* \in X$.

Observe $d(\overset{\varphi(x_n)}{x_{n+1}}, \overset{\varphi(x_{n-1})}{x_n}) = d(\varphi(x_n), \varphi(x_{n-1})) \le c\, d(x_n, x_{n-1})$ by contraction cond

then ($\star$) $d(x_{n+1}, x_n) \le c\, d(x_n, x_{n-1}) \le \dots \le c^n d(x_1, x_0)$

For Cauch seq, let large pos integers $n < m$

$$d(x_m, x_n) \underset{\text{Triang ineq}}{\le} \underbrace{d(x_{n+1}, x_n)}_{\le c^n d(x_1,x_0)} + \underbrace{d(x_{n+1}, x_{n+2})}_{c^{n+1} d(x_1,x_0)} + \dots + \underbrace{d(x_m, x_{m-1})}_{c^{m-1} d(x_1,x_0)} \quad \text{by } (\star)$$

$$\le \underbrace{(c^n + c^{n+1} + \dots + c^{m-1})}_{(1 + c + \dots + c^{(m-1)-n})c^n} d(x_1, x_0) \qquad p = m-n-1$$

Finite Geo sums:

$S = 1 + a + a^2 + \dots + a^p$

$\Rightarrow aS = a + a^2 + \dots + a^{p+1}$

Subtract $S - aS = 1 - a^{p+1}$

$S = \dfrac{1 - a^{p+1}}{1-a}$

Finite geo sum $\left[\dfrac{1 - c^{p+1}}{1-c}\right] c^n = \left[\dfrac{1 - c^{m-n}}{1-c}\right] c^n$

$$\Rightarrow d(x_m, x_n) \le \left[\frac{1 - c^{m-n}}{1-c}\right] c^n d(x_1, x_0) < \left[\frac{1}{1-c}\right] d(x_1, x_0)\, c^n$$

Since $c<1$, c^n arb small for n large enough $\Rightarrow (x_n)$ is Cauchy seq.

Since X complete, $\exists\, x^* \in X$ such than $(x_n) \to x^*$

▷ Is x^* unique? Yes.

Suppose for some initial pts x_0, y_0 $(\varphi^n(x_0)) \to x^*$ while $(\varphi^n(y_0)) \to y^*$ $\quad \varphi(x^*) = x^*$ $\quad \varphi(y^*) = y^*$

By the contraction cond $d(\underset{x^*}{\varphi(x^*)}, \underset{y^*}{\varphi(y^*)}) \le c\, d(x^*, y^*)$

$\Rightarrow d(x^*, y^*) \le c\, d(x^*, y^*)$ only possible if $d(x^*, y^*) = 0$
$\Rightarrow x^* = y^*$.

▷ φ is Unif Cont in X

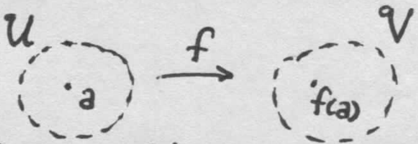
Given $\varepsilon > 0$, $\exists\, \delta > 0$ (indep of x,y) $\ni$ for any $x,y \in X$
$d(\varphi(x), \varphi(y)) < \varepsilon$ if $d(x,y) < \delta$

By contraction cond $d(\varphi(x), \varphi(y)) \le c\, d(x,y) < c\delta$ so take $\delta = \varepsilon/c$

▷ $\varphi(x^*) = \varphi(\lim x_n) = \lim \varphi(x_n) = \lim x_{n+1} = x^*$

$\boxed{\text{QED}}$

Inv Fcn Thm



Thm X, Y Banach
$f : \underset{X}{\mathcal{O}} \longrightarrow Y$ C^N smooth
$A^{-1} := [Df_a]^{-1}$ exists for one $a \in \mathcal{O}$
(In fin dim case, this forces $\dim X = \dim Y$)

$\Rightarrow$ $f : \mathcal{U} \to \mathcal{V}$ is a C^N diffeo
In more detail: • f^{-1} exists near a
• $D(f^{-1})_{f(x)}$ exists in $\mathcal{V}$
• $D(f^{-1})_{f(x)} = [Df_x]^{-1}$ $x \in \mathcal{U}$
• f^{-1} is C^N if f C^N

Df_a dominates the behavior of f near a

Pf. **Step 1** Establish f is One-to-One in a nbhd of a [i.e. f^{-1} exists there]

Since Df_x is a cont fcn of x, Given $\lambda > 0$ $\exists\, \delta > 0$ $\ni$
$x \in \mathring{B}(a,\delta) \Rightarrow |Df_x - Df_a| < \lambda$ Take $\lambda := \dfrac{1}{2\|\cancel{Df_a}\, A^{-1}\|}$ i.e. $\|\cancel{Df_a^{-1}}\, A^{-1}\| = \dfrac{1}{2\lambda}$

A = fixed map Df_a
$A^{-1} = [Df_a]^{-1}$

Then for the associated δ $\mathcal{U} := \mathring{B}(a,\delta)$

▷ For each $y \in Y$, define maps $\varphi_y : \mathcal{U} \longrightarrow X$
$x \longmapsto x + Df_a^{-1}(y - f(x))$

⊛ $\boxed{\text{observe } f(x) = y \Leftrightarrow \varphi_y(x) = x \text{ F.P.}}$

Fix y
▷ Show φ_y satisfies Contraction cond: $\|\varphi_y(x_1) - \varphi_y(x_2)\| \le \frac{1}{2}\|x_1 - x_2\|$

From MVT (Avez DC ch 1 sheet ⓒ): $\|\varphi_y(x_2) - \varphi_y(x_1)\| \le \|D(\varphi_y)_x\|_{op}\|x_2 - x_1\|$

$D(\varphi_y)_x = I - Df_a^{-1}Df_x = Df_a^{-1}[Df_a - Df_x]$

then $\|D(\varphi_y)_x\| \le \|Df_a^{-1}\| . \|Df_a - Df_x\| < \frac{1}{2\lambda}\lambda = \frac{1}{2}$ since $x \in \mathcal{U} = \mathring{B}(a,\delta)$

▷ φ_y can have at most one FP in $\mathcal{U}$ because if $\varphi(x_1) = x_1$ and $\varphi(x_2) = x_2$ $x_1 \ne x_2$
$\|\underset{x_1}{\varphi(x_1)} - \underset{x_2}{\varphi(x_2)}\| \le \frac{1}{2}\|x_1 - x_2\| \Rightarrow \|x_1 - x_2\| \le \frac{1}{2}\|x_1 - x_2\|$ This is only possible if $\|x_1 - x_2\| = 0$
i.e. $x_1 = x_2$.

We have now show f is One-to-One in $\mathcal{U}$ because if $f(x_1) = y$ and $f(x_2) = y$ by ⊛
then $\varphi(x_1) = x_1$ and $\varphi(x_2) = x_2$ which we have just shown forces $x_1 = x_2$

Step 2 Define $\mathcal{V} = f(\mathcal{U})$. Show $\mathcal{V}$ is open in Y by showing every pt is an interior pt.

choose any $y_0 \in f(\mathcal{U})$. Then $\exists\, x_0 \ni f(x_0) = y_0$
Since $\mathcal{U}$ is open, $\exists\, r > 0 \ni \bar{B}(x_0, r) \subset \mathcal{U}$.

We will show y_0 is an interior pt of $\mathcal{V}$ by showing <u>every pt y</u> in $B(y_0, \lambda r)$
is also the image of some x: $f(x) = y$.

Let y be any pt in $B(y_0, \lambda r)$

▷ observe $\varphi_y : \overline{B}(x_0, r) \longrightarrow \overline{B}(x_0, r)$ because, for any $x \in \overline{B}$, $\|\varphi_y(x) - x_0\| < r$:

$$\varphi_y(x) - x_0 = \varphi_y(x) - \varphi_y(x_0) + \varphi_y(x_0) - x_0$$

$$\|\varphi_y(x) - x_0\| \le \underbrace{\|\varphi_y(x) - \varphi_y(x_0)\|}_{\substack{\le \frac{1}{2}\|x - x_0\| \\ < \frac{1}{2} r}} + \underbrace{\|\varphi_y(x_0) - x_0\|}$$

$$= \|\cancel{x_0} - Df_a^{-1}(y - f(x_0)) - \cancel{x_0}\|$$
$$= \|Df_a^{-1}(y - y_0)\|$$
$$\le \|Df_a^{-1}\|_{op} \|y - y_0\|$$
$$< \frac{1}{2\lambda} \quad \lambda r \quad = \frac{1}{2} r$$

so $\varphi_y(x) \in \overline{B}$

▷ Since $\overline{B}$ is a complete metric sp, we can apply Contraction Map FP Thm
$\Rightarrow \exists! \; x_y^* \in \overline{B} \ni \varphi_y(x_y^*) = x_y^*$ and by (A) $f(x_y^*) = y$ so $B(y_0, \lambda r)$ contains only pts of $f(\mathcal{U})$

Step 3 we know $f: \overset{\text{open}}{\mathcal{U}} \to \overset{\text{open}}{\mathcal{V}}$ and f^{-1} exists there. Now we must show f^{-1} difb.

▷ Show $[Df_x]^{-1}$ exists and is bdd.
we have $\|Df_x - Df_a\| < \lambda$ where $\lambda = \frac{1}{2\|A^{-1}\|}$ $A := Df_a$
Thus $\|Df_x - Df_a\| < \lambda < 2\lambda = \frac{1}{\|A^{-1}\|}$ i.e. $\|Df_x - Df_a\| \, \|A^{-1}\| < 1$ see sheet

we can apply Thm Invertible Maps form an Open Set and get:
linear map $T := [Df_x]^{-1}$ exists and $\|T\| \le \dfrac{\|A^{-1}\|}{1 - \|A^{-1}\| \, \|Df_x - Df_a\|}$

▷ But is $[Df_x]^{-1} = D(f^{-1})_{f(x)}$? Do we even know RHS exists?
Lets denote $g := f^{-1}$. The defining cond for Dg_y to exist is $\exists$ linear map $L \ni$

$$\lim_{k \to 0} \frac{\|g(y+k) - g(y) - Lk\|}{\|k\|} = 0$$

we will plug in T for L and thus show $L = Dg_y$ exists and $Dg_y = T$

Choose $y \in \mathcal{V}$ and k small enough such that $y + k \in \mathcal{V}$
Then $\exists$ x, $x+h$ where $f(x) = y$ and $f(x+h) = y+k$ NOTE: $f(x+h) - f(x) = y + k - y = k$

$$g(y+k) - g(y) - Tk = x + h - x - Tk = h - Tk$$
$$= TT^{-1}h - Tk = T[\underset{Df_x h}{T^{-1}h} - \underset{(f(x+h) - f(x))}{k}]$$
$$= -T[f(x+h) - f(x) - Df_x h]$$

cont'd →

Take norms and divide by $\|k\|$:

$$\frac{\|g(y+k)-g(y)-Tk\|}{\|k\|} \leq \frac{\|T\|\,\|f(x+h)-f(x)-Df_x h\|}{\|k\|} \leq \frac{\|T\|}{\lambda}\left[\frac{\|f(x+h)-f(x)-Df_x h\|}{\|h\|}\right]$$

CLAIM: $\|h\| \leq \frac{1}{\lambda}\|k\|$

Once we establish the claim,
Then $|k| \to 0 \Rightarrow |h| \to 0$ Then the RHS of above ineq $\to 0$ and thus the LHS also.
Thus g is dif'b (hence Cont) and $Dg_y = T$ i.e. $D(f^{-1})_{f(x)} = [Df_x]^{-1}$

Δ Establish Claim: $\|h\| \leq \frac{1}{\lambda}\|k\|$

For any $z \in \mathcal{V}$ $\quad \varphi_z(x+h) = x+h + A^{-1}[z - f(x+h)]$
$\quad \varphi_z(x) = x + A^{-1}[z - f(x)]$

$$\varphi_z(x+h) - \varphi_z(x) = h + A^{-1}[\underbrace{-f(x+h)}_{-(y+k)} + \underbrace{f(x)}_{y}] = h - A^{-1}k$$

Thus $\|\varphi_z(x+h) - \varphi_z(x)\| = \|h - A^{-1}k\|$

We also have the contraction Cond: $\|\varphi_z(x+h) - \varphi_z(x)\| \leq \frac{1}{2}\|x+h-x\| = \frac{1}{2}\|h\|$

$\Rightarrow \|h - A^{-1}k\| \leq \frac{1}{2}\|h\|$

Reverse Triang: $\|h\| - \|A^{-1}k\| \leq \frac{1}{2}\|h\| \Rightarrow -\|A^{-1}k\| \leq -\frac{1}{2}\|h\| \Rightarrow \|h\| \leq 2\|A^{-1}k\| \leq 2\|A^{-1}\|\,\|k\|$

$= \cancel{2}\left(\frac{1}{\cancel{2}\lambda}\right)\|k\|$

$= \frac{1}{\lambda}\|k\|$

$\boxed{\text{END CLAIM}}$

$\overset{h}{b}(x,y) = x^{-1}$ $\quad Db$ $\quad$ 12/5/2023

$\overset{hu^{-1}k}{b}(u^{-1}, u^{-1})$ $\qquad \overset{ku^{-1}h}{b}_{u^{-1}u^{-1}}$ $\qquad u: E \to F$, $J: GL_{EF} \to GL_{EF}$

$$\left.\frac{\partial}{\partial s}\frac{\partial J}{\partial t}\right|_{\substack{t=0\\ s=0}} = \underbrace{u^{-1}hu^{-1}}ku^{-1} + u^{-1}k\underbrace{u^{-1}hu^{-1}} = D^2J_u(h,k)$$

$$\overset{k}{b}(u^{-1}hu^{-1}, u^{-1}) + \overset{k}{b}(u^{-1}, u^{-1}hu^{-1})$$

$$DJ: u \overset{J}{\longmapsto} u^{-1} \overset{\mathrm{I}}{\longmapsto} (u^{-1}, u^{-1}) \overset{b}{\longmapsto} f_{u^{-1}u^{-1}}(\cdot)$$

or I could say

$$DJ^h: u \longrightarrow u^{-1} \longrightarrow (u^{-1}, u^{-1}) \overset{\overset{h}{b}}{\longmapsto} f_{u^{-1}u^{-1}}(h) = -u^{-1}hu^{-1}$$

so we have $\overset{p}{b}(x,y) = xpy$

$$D\overset{p}{b}_{(x_1,x_2)}(h_1 h_2) = \overset{p}{b}(x_1, h_2) + \overset{p}{b}(h_1, x_2)$$
$$= x_1 p h_2 + h_1 p x_2$$

and

$$D\overset{p}{b}_{xx}((h_1 h_2)) = x p h_2 + h_1 p x = \overset{p}{b}(x, h_2) + \overset{p}{b}(h_1, x)$$

$$DJ = B \circ \mathrm{I} \circ J$$
$$DJ_u(h) = -u^{-1}hu^{-1} = -\overset{h}{b}_{(u^{-1},u^{-1})}$$

$$D^2J = D(DJ^h) = D(B \circ \mathrm{I} \circ J)^{(k)} = DB \cdot (DI \cdot DJ)(k)$$
$$= D(\overset{h}{b}(u^{-1}, u^{-1}))$$

$$DB_g(k) = B_g(g, k_1) + B(k_2, g)$$
$$g k_1 g + g k_2 g$$

$$DB_{gg}((k\,k))$$

(g, g)

$$D\overset{k}{b}_{u^{-1}u^{-1}}((u^{-1}hu^{-1}, u^{-1}hu^{-1}))$$

$$DB_{\mathrm{I}J(u)}(\underbrace{D\mathrm{I}_{J(u)}}_{\mathrm{I}}(DJ_u(\cdot)))$$

Is that it??

$$D\overset{k}{b}_{(\mathrm{I}\circ J)u}(\mathrm{I}(DJ_u(h)))$$

$$DB_{\mathrm{I}Ju}(\mathrm{I}(DJ_u(\cdot)) \quad h$$

$$DB_{u^{-1},u^{-1}}((DJ_u(h), DJ_u(h)))(k)$$

And $\overset{h}{b}(x,y)$ is really $\overset{h}{b}(-x,y)$

$$x \longmapsto b(-x, x)$$