$$f : \overbrace{E_1 \oplus E_2 \oplus \cdots \oplus E}^{E} \longrightarrow \overset{\text{nes}}{F}$$
$$\langle x_1, \dots x_n \rangle \longmapsto f(x)$$

ch 1.2.9 Maps defined ON a direct sum

big idea $\boxed{\text{Partial Derivs}}$

We are going to show $Df_a(h) = D_1 f_a(h_1) + \dots + D_n f_a(h_n)$ where $a = \langle a_1, \dots a_n \rangle$, $h = \langle h_1, \dots h_n \rangle$

[I could upper index, but generally only do that in $\mathbb{R}^n$.]

so fix a

Define a couple of helper maps:

$i_r^{(0)} : E_r \longrightarrow E$ Inclusion in 0 vector of E — abbreviate i_r

$x_r \longmapsto \langle 0 \dots x_r \dots 0 \rangle$ linear map so $D(i_r)_{a_r}(h_r) = i_r(h_r)$

$J_r^a : E_r \longrightarrow E$ subs x_r in a at position r

$x_r \longmapsto \langle a_1, \dots x_r \dots a_n \rangle$

so $J_r^a(x_r) = a + i_r(x_r - a_r) = a + i_r(x_r) - i_r(a_r)$ smooth in x_r

$J_r^a(a_r) = a$ and $D(J_r^a)_{a_r}(h_r) = D(i_r)_{a_r}(h_r) = i_r(h_r)$ ⊛

$$(f \circ J_r^a) : E_r \longrightarrow E_1 \oplus \cdots \oplus E_n \xrightarrow{f} F$$
$$x_r \longmapsto \langle a_1, \dots x_r \dots a_n \rangle \longmapsto f(a_1 \dots x_r \dots a_n)$$

$\boxed{\text{DEFINE } D_r f_a(h_r)} := D(f \circ J_r^a)_{a_r}(h_r)$ This deriv exists since everything is smooth

we can now use chain rule

$$= Df_{\underbrace{J_r^a(a_r)}_{a}} \underbrace{D(J_r^a)_{a_r}(h_r)}_{i_r(h_r) \text{ by ⊛}}$$

Thm Df_a exists $\Rightarrow$ $Df_a(h) = D_1 f_a(h_1) + \dots + D_n f_a(h_n)$ (partials exist at a)

Df_a exists and is cont in a $\Leftarrow$ all Partials exist and are continuous at a in nbhd of a (this is ch 2 Thm 2.5 p. 14)

(f is C^1 at a

Pf ($\Rightarrow$) Note identity $I(x) = x = \sum_{r=1}^{n} (i_r \circ \pi_r)(x)$ where π_r is std projection

e.g. $\langle x_1\ 0\ 0 \rangle + \langle 0\ x_2\ 0 \rangle + \langle 0\ 0\ x_3 \rangle = \langle x_1\ x_2\ x_3 \rangle = x$

so $f(x) = f(I(x)) = f \circ \left(\sum_r (i_r \circ \pi_r)(x) \right)$

$$Df_a(h) = Df_{\underbrace{\sum_r (i_r \circ \pi_r)(a)}_{a}} \left(\sum D(i_r)_{\underbrace{\pi_r(a)}_{a_r}} \underbrace{D(\pi_r)_a(h)}_{h_r} \right)$$

Df_a linear

$$= \sum_r Df_a \big(\underbrace{i_r(h_r)}_{D(J_r^a)_{a_r}(h) \text{ by ⊛}} \big)$$

$$= \sum_r D(f \circ J_r^a)_{a_r}(h)$$

by def

$$= \sum_r (D_r f)_a (h_r) \quad \square$$

cont'd ⟶

(From p. 14)

($\Leftarrow$) We know $Df_a(h) = \sum_k D_k f_a(h_k)$ so it is enough to show Df_a exists.

The diff quotient is $\dfrac{\| f(a+h) - f(a) - Df_a(h) \|}{\|h\|}$ so define $g(x) := f(x) - f(a) - \sum_k D_k f_a(x_k - a_k)$

Step 1 Define g

$x := a+h$

$x - a = h$

observe $g(a) = 0$

Then for any k,

This is Cont as fcn of x $\longrightarrow$ $D_k g_x = D_k f_x - D_k f_a$ $\qquad D_k f_a(x_k)$ is linear in x_k

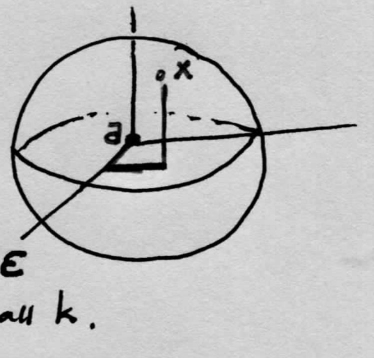
We are going to do basically the same arg that Rudin POMA p.219 does for $\mathbb{R}^n$.

[M&T ch 2 write-up sheet (3)]

We will apply MVT to g on each path segment.

Step 2 Bound g

$\|D_k g_x\| = \|D_k f_x - D_k f_a\|$ and all partials are cont at a.



Thus given $\varepsilon > 0$, for each k $\exists\, r_k \ni$

$$x \in \mathring{B}(a, r_k) \Rightarrow \|D_k g_x\| = \|D_k f_x - D_k f_a\| < \varepsilon$$

Let $r := \min\{r_1 \dots r_n\}$ Then $\sup_{x \in \overline{B}(a,r)} \|D_k g_x\| \leq \varepsilon$ for all k.

Step 3 Trickery with MVT

n=3 case

$$g(x) = g(x) - \overset{0}{\overline{g(a)}}$$

add & subtr $\qquad$ add & subtr

$$= g(x_1 a_2 a_3) - g(a_1 a_2 a_3) + g(x_1 x_2 a_3) - g(x_1 a_2 a_3) + g(x_1 x_2 x_3) - g(x_1 x_2 a_3)$$

$$\leq \underbrace{\|g(x_1 a_2 a_3) - g(a_1 a_2 a_3)\|}_{\text{Segment 1}} + \underbrace{\|g(x_1 x_2 a_3) - g(x_1 a_2 a_3)\|}_{\text{Segment 2}} + \underbrace{\|g(x_1 x_2 x_3) - g(x_1 x_2 a_3)\|}_{\text{Segment 3}}$$

apply MVT Bounds on each segment

For example, segment 1 : $\|g(x_1 a_2 a_3) - g(a_1 a_2 a_3)\| \leq \overbrace{\sup_{x \in \overline{B}(a,r)} \|D_1 g_x\|}^{\varepsilon} \|x_1 - a_1\|_{E_1} \leq \varepsilon \|x - a\|_E$

$\|\langle x_1, x_2 \rangle\|_{E_1 \times E_2} = \|x_1\|_{E_1} + \|x_2\|_{E_2}$

Thus we see $\|g(x)\| \leq \sum_{k=1}^{n} \varepsilon \|x - a\| = n\varepsilon \|x - a\|$

Step 4

$\dfrac{\|g(x)\|}{\|x-a\|} \leq n\varepsilon$ $\quad$ rewriting $h = x - a$ $\quad$ $\dfrac{\|f(a+h) - f(a) - \sum_k D_k f_a(h_k)\|}{\|h\|} \leq n\varepsilon$

$\|h\| < r$

This shows Df_a exists, and by uniqueness $Df_a(h) = \sum_k D_k f_a(h_k)$ $\square$

Let's illustrate the important case $f: \mathbb{R}^n \to \mathbb{R}^m$ as discussed in Rudin POMA p.215 and written up in my M&T ch 2 sheets (where I focus on $\mathbb{R}^n$)

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad \langle x_1 \ldots x_n \rangle \longmapsto \begin{bmatrix} f^1(x) \\ \vdots \\ f^m(x) \end{bmatrix}$$

Define partial $\frac{\partial f^i}{\partial x_j}(x) = D_j f^i(x) = \lim_{t \to 0} \frac{f^i(x+t\hat{e}_j) - f(x)}{t}$

$$\boxed{\underline{\text{Thm}}\ f \text{ difb at } x \Rightarrow \text{all partials } \frac{\partial f^i}{\partial x_j} \text{ exist at } x \text{ (not nec cont) and } Df_x = \left[\frac{\partial f^i}{\partial x_j}\right] \text{ matrix}}$$

Pf. f difb $\Rightarrow \exists!$ linear map $A_x : \mathbb{R}^n \to \mathbb{R}^m \ni f(x+h) - f(x) = A_x h + o(h)$ ⊛

Fix j $1 \le j \le n$ so $\hat{e}_j$ is a fixed std basis vector.

Let $h = t\hat{e}_j$

$$⊛ \Rightarrow \lim_{t \to 0} \frac{\vec{f}(x+te_j) - \vec{f}(x)}{t} = \frac{1}{\cancel{t}} A(\cancel{t}e_j) + \lim_{t \to 0} \cancel{\frac{o(\cancel{t}e_j)}{t}}^{\,0}$$

$$\lim_{t \to 0} \begin{bmatrix} \frac{f^{1}(x+te_j) - f^{1}(x)}{t} \\ \vdots \\ \frac{f^{m}(x+te_j) - f^{m}(x)}{t} \end{bmatrix} = Ae_j = \begin{bmatrix} a_j^{1} \\ \vdots \\ a_j^{m} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial f^{1}}{\partial x_j}(x) \\ \vdots \\ \frac{\partial f^{m}}{\partial x_j}(x) \end{bmatrix} = \begin{bmatrix} a_j^{1} \\ \vdots \\ a_j^{m} \end{bmatrix}$$

This is the j^{th} col
we get all cols of $Df_x = A$
by going thru basis
vectors $e_1, e_2, \ldots e_n$

□