Ch 4.1 as before, consider $f: \mathcal{U} \xrightarrow{E} F$ then $Df_a : E \to F$ Bdd Linear

$Df_a \in \mathcal{L}(E \to F)$ and

$Df : \mathcal{U} \to \mathcal{L}(E \to F)$
$x \longmapsto Df_x(\cdot)$

Then $D^2 f_a \in \mathcal{L}(E \to \mathcal{L}(E \to F))$ not F!

and $\mathcal{L}(E \to \mathcal{L}(E \to F)) \cong \mathcal{L}(E \times E \to F)$ ← Cannonical isomorphism p.151

Fix $h \in E$ $\quad F^h(x) := Df_x(h)$

Then we seek linear map $A^h_x(\cdot)$ such than $\dfrac{\| F^h(x+k) - F^h(x) - A^h_x(k)\|_{\mathcal{L}(E\to F)}}{\|k\|_E} \to 0$

$A^h_{(\cdot)}$ is linear in h already, now it is linear in new arg k.

and $\|Q\|_{\mathcal{L}(E\to F)}$ is $\sup_{\|v\|=1} \|Qv\|_F$

i.e. $\dfrac{\| Df_{x+k}(h) - Df_x(h) - A_x(k;h)\|_F}{\|k\|_E} \to 0$

$D^2 f_x(h,k)$

$A_x(k;h)$ is bi-linear.

Ch 4.1.2 Calculation of $D^2 f_a$

From Ch 1 p.1-2 $\dfrac{d}{dt} f(x+th)\Big|_{t=0} \overset{def}{=} \lim_{t\to 0} \dfrac{f(x+th)-f(x)}{t} = \delta f_x(h) = Df_x(h)$

These values are equal

Now for $F^h(x) := Df_x(h)$

$$\delta F^h_x(k) = \frac{d}{ds} F^h(x+sk)\Big|_{s=0} = \frac{d}{ds}\left(Df_{x+sk}(h)\right)\Big|_{s=0}$$
$$= \frac{d}{ds}\left(\frac{d}{dt} f(x+th+sk)\Big|_{t=0}\right)\Big|_{s=0}$$

So this method would give us the value ptwise $D^2 f_x(h,k)$

I'm not sure this method is useful, so here are some other calculations

(1) f is <u>linear</u> (continuous i.e. Bdd Linear) $\quad f: E \to F$
$x \longmapsto Lx$ where linear map L does not depend on x

Then from ch 1 $Df_x = L$ and since there is no x dependence $D^n f_x \equiv 0 \ \forall n \geq 2$

(2) f is <u>bilinear</u> (continuous) $\quad b : E_1 \oplus E_2 \to F \quad$ For $x = \langle \vec{x}_1, \vec{x}_2 \rangle$ "f" $\mathcal{B}(x) := b(x_1, x_2)$

M&T Ch 4.2 write-up

Then from ch 1 $\underbrace{D\mathcal{B}_x(h)}_{F^h(x)} = b(x_1, h_2) + b(h_1, x_2)$

To find $D^2\mathcal{B}_x$ compute $F^h(x+k) - F^h(x) = b(x_1+k_1, h_2) + b(h_1, x_2+k_2) - [b(x_1,h_2) + b(h_1,x_2)]$
$= \cancel{b(x_1,h_2)} + b(k_1,h_2) + \cancel{b(h_1,x_2)} + b(h_1,k_2) - \cancel{b(x_1,h_2)} - \cancel{b(h_1,x_2)} = \boxed{b(k_1,h_2) + b(h_1,k_2)}$

Linear in k; no dependence on x so $D^3\mathcal{B} \equiv 0$, etc...

Now specialize to $\mathbb{R}^n$ and b symm: $b(x,y) = b(y,x) \Rightarrow x^TAy = y^TAx \Rightarrow A = A^T$

quadratic: $f(x) = b(x,x) = x^TAx$

$Df_x(h) = 2x^TAh$

$D^2f_x(h,h) = b(h,h) + b(h,h) = 2b(h,h) = 2h^TAh$

So $D^2 f_x(\cdot,\cdot) = 2A$

□

Avez DC ch 4 7/13/2022

Let's show this 2^{nd} deriv calculation for $f: \mathbb{R}^n \to \mathbb{R}$

[Avez doesn't do this until much later in Chapter, and no details]

Recall: $f: \mathbb{R}^3 \to \mathbb{R}$ why is $Df_x = [D_1 f \ \ D_2 f \ \ D_3 f]$?
$[f_x \ \ f_y \ \ f_z]$

f difb $\Rightarrow \exists\, A_x: \mathbb{R}^3 \to \mathbb{R} \ni f(x+h) - f(x) = A_x h + o(h)$

Then for any basis vector $\hat{e}_j$ let $h = t\hat{e}_j$ $\displaystyle\lim_{t\to 0} \frac{1}{t}[f(x+h) - f(x)] = \lim_{t\to 0} \frac{A_x(\cancel{t}e_j)}{\cancel{t}} + \lim \frac{\cancel{o(te_j)}}{t}$ $\ 0$

say $j=2$ $\hat{e}_2$ $\displaystyle\lim_{t\to 0} \frac{f(x, y+te_2, z) - f(x,y,z)}{t} = [a_1^{(1)} \ a_2^{(1)} \ a_3^{(1)}]\begin{bmatrix}0\\1\\0\end{bmatrix}$

$f_y(x,y,z) = a_2^{(1)}$

Thus we see $[D_1 f \ \ D_2 f \ \ D_3 f] = [a_1^{(1)} \ a_2^{(1)} \ a_3^{(1)}]$ and A is unique, so we know this is it.

▷ Now lets apply this to 2^{nd} deriv:

Fix $h = \begin{bmatrix}h^1\\h^2\\h^3\end{bmatrix}$ Let $F^h(x) := Df_x(h) = \underbrace{[f_x(x) \ f_y(x) \ f_z(x)]}_{\text{new "A"}}\begin{bmatrix}h^{(1)}\\h^{(2)}\\h^{(3)}\end{bmatrix} = \sum^3 h_i D_i f(x) \in \mathbb{R}$

$F^h: \mathbb{R}^3 \to \mathbb{R}$
$x \mapsto \sum h_i D_i f(x)$

Thus $A_x^h: \mathbb{R}^3 \to \mathbb{R}$

$$A_x^h(k) = [a_1^h(x) \ a_2^h(x) \ a_3^h(x)]\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = [D_1 F^h(x) \ \ D_2 F^h(x) \ \ D_3 F^h(x)]\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}$$

$$= \left[D_1\left(\sum h_i D_i f(x)\right) \ \ D_2\left(\sum h_i D_i f\right) \ \ D_3\left(\sum h_i D_i f\right)\right]\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}$$

$$= \left[\sum^3 h_i D_1 D_i f \ \ \ \sum h_i D_2 D_i f \ \ \ \sum h_i D_3 D_i f\right]\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix}$$

$$= \sum^3 h_i [D_1 D_i f \ \ D_2 D_i f \ \ D_3 D_i f]\begin{bmatrix}k\end{bmatrix} = \left(h_1[D_{11} \ D_{21} \ D_{31}] + h_2[D_{12} \ D_{22} \ D_{32}] + h_3[D_{13} \ D_{23} \ D_{33}]\right)[k]$$

$$= [h_1 \ h_2 \ h_3]\begin{bmatrix}D_{11}f & D_{21}f & D_{31}f\\ D_{12}f & D_{22}f & D_{32}f\\ D_{13}f & D_{23}f & D_{33}f\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = D^2 f_x(h,k)$$

□

Did I accidentally end up with transpose of matrix of partials? Usually no problem when $D_{ij} = D_{ji} f$

If we had $f: \mathbb{R}^n \to \mathbb{R}^m$, then this calculation would apply to each component $f^{(i)}$ and we'd end up with a stack of m of these

Say $n=2$ $m=2$

$$D^2 f_x(h,k) = \begin{bmatrix} [h_1 \ h_2]\begin{bmatrix}D_{11}f^{(1)} & D_{21}f^{(1)}\\ D_{12}f^{(1)} & D_{22}f^{(1)}\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix} \\ \\ [h_1 \ h_2]\begin{bmatrix}D_{11}f^{(2)} & D_{21}f^{(2)}\\ D_{12}f^{(2)} & D_{22}f^{(2)}\end{bmatrix}\begin{bmatrix}k_1\\k_2\end{bmatrix}\end{bmatrix}$$

ch 4.1.3. **Schwarz Symmetry of 2^{nd} Deriv Thm**

$f: \underset{E}{\mathcal{U}} \longrightarrow F$ 2 times difb $\Rightarrow$ • $\dfrac{\| f(a+u+v) - f(a+u) - f(a+v) + f(a) - D^2 f_a(u,v) \|}{(\|u\| + \|v\|)^2} \to 0$ as $u \to 0$, $v \to 0$

I write this C^{1+d}

• $D^2 f_a(\cdot,\cdot)$ is bilinear and symm

$D^2 f_a(u,v) = D^2 f_a(v,u)$

Pf.

Avez ch 4.2

Skipping 4.1.4 $C^n(\mathcal{U} \to F)$

4.1.5 Generalized Schwarz f is C^n then $D^n f_a(h_1 \dots h_n) = D^n f_a(h_{\sigma(1)} \dots h_{\sigma(n)})$ where σ is any perm of $\{1,\dots,n\}$

4.1.6 Higher order Gateaux $\delta^n f_a = D^n f_a$

Ch 4.2 Rules for calculation

we already did linear and bilinear f

4.2.3 Leibniz' Rule Better: "Bilinear Prod Rule"

Recall from ch 1: define $\boxed{fg} : E \xrightarrow{\lambda} F_1 \times F_2 \xrightarrow{b} G$

$x \longmapsto \langle f(x), g(x) \rangle \longmapsto b(f(x), g(x))$

Thm $D\boxed{fg}_x(h) = b(Df_x(h), g(x)) + b(f(x), Dg_x(h))$ and f, g $C^1 \Rightarrow \boxed{fg}$ C^1

pf. By chain rule $D\boxed{fg}_x(\cdot) = Db_{\lambda(x)} D\lambda_x(\cdot)$ where $D\lambda_x(h) = (Df_x(h), Dg_x(h))$

$= Db_{(f,g)}(Df_x(h), Dg_x(h))$

$= b(Df_x(h), g(x)) + b(f(x), Dg_x(h))$

This is obviously Continuous in x, so $\boxed{fg}$ is C^1

▷ Now if f, g are 2 times difb $\Rightarrow \boxed{fg}$ is 2-times difb (or replace by C^2 and that is also true)

This follows because $D\boxed{fg}_x(h) = b(F^h(x), g(x)) + b(f(x), G^h(x))$

$= \boxed{F^h g}(x) + \boxed{f G^h}(x)$

so then $D^2\boxed{fg}_x(h,k) = D^1\boxed{F^h g}_x(k) + D^1\boxed{fG^h}_x(k)$

Thus, inductively, $f, g \in C^k \Rightarrow \boxed{fg} \in C^k$

Skipping 4.2.4 Higher-order derivs of a product: $(fg)^{(n)} = \sum_{q=0}^{n} \binom{n}{q} f^{(n-q)} g^{(q)}$

Symmetrization operator $\mathrm{Sym}(A)(h_1 \dots h_n) := \frac{1}{n!} \sum_{\substack{\text{all perms} \\ \sigma}} A(h_{\sigma(1)}, \dots, h_{\sigma(n)})$

▷ ch 4.2.5 Derivs of Composite Maps

I am skipping a lot because I am interested in $D^2(f \circ g)_x(h)(k)$

GOOD STUFF

7/17/2022

$X \xrightarrow{g} Y \xrightarrow{f} Z$

x

h, k

Given a composite map $f \circ g$ we can compute $D(f \circ g)_x$ by the chain rule $Df_{g(x)} Dg_x(\cdot)$

How do we compute $D^2(f \circ g)_x = D(Df_{g(x)} Dg_x(\cdot))_x(\cdot)$?

We have Leibniz bilinear product rule:

For $f \star g : E \longrightarrow F_1 \times F_2 \longrightarrow G$ / "b" is the "$\star$"

$x \longmapsto \langle f(x), g(x) \rangle \longmapsto b(f(x), g(x))$

Then $D(f \star g)_x(h) = b(Df_x(h), g(x)) + b(f(x), Dg_x(h)) = Df_x(h) \circledast g(x) + f(x) \circledast Dg_x(h)$

Now let "$\star$" be matrix multiplication $A\vec{q} = A(x)\vec{q}(x)$

$$D(A \star q)_x(h) = DA_x(h) \circledast q(x) + A(x) \circledast Dq_x(h)$$

Now $A(p(x)) q(x) = (A \circ p)(x) \circledast q(x)$

$$D((A \circ p) \star q)_x(h) = D(A \circ p)_x(h) \star q(x) + (A \circ p)(x) \star Dq_x(h)$$

chain rule

$$= DA_{p(x)} Dp_x(h) \circledast q(x) + A_{p(x)} \circledast Dq_x(h)$$

Now for the last step: Let $(A \circ p)(x) = Df_{g(x)}(\cdot)$

$$q(x) = G^h(x) = Dg_x(h)$$

Then $D(Df_{g(x)} Dg_x(h))_x(k) = D(Df_{g(x)})_{g(x)} Dg_x(k) \star Dg_x(h) + Df_{g(x)}(\underbrace{DG^h_x(k)})$

$D^2 g_x(h,k)$ by def

$$= D^2 f_{g(x)}(Dg_x(k), \cdot) \star Dg_x(h) +$$

$$\boxed{= D^2 f_{g(x)}(Dg_x(k), Dg_x(h)) + Df_{g(x)}(D^2 g_x(h,k))}$$

I am not sure about the order but it wont matter since $D^2 f$ symm.

By def

If we let $G^h(x) = Dg_x(h)$

then

$D(G^h)_x(k) = D^2 g_x(h,k)$

$D(Dg_x(h))_x(k)$

$D(F^h \circ g$

Lets do an example $\begin{bmatrix} a(p_1(x,y,z)) & b(p(x)) & c(p(x)) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$

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2

Let's recap the examples

① Aq $\quad D(A \star q)_x(h) = DA_x(h) \circledast q(x) + A_{(x)} \star Dq_x(h)$

$$\mathbb{R} \longrightarrow (\mathbb{R}^3)^* \times \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$t \longmapsto \langle A(t)\ q(t) \rangle \longmapsto b(A, q) = [a(t)\ b(t)\ c(t)] \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

and we can verify this just by multiplying this out and using ordinary calculus

Here we don't really 'need' h to get the pieces to fit together, but we do in the other cases

$$D(Aq)_t(h) = h[\dot{a}\ \dot{b}\ \dot{c}]\begin{bmatrix} q \end{bmatrix} + [a\ b\ c]\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}[h]$$

To be analogous

what about $A_{(x,y,z)}\vec{q}(x,y,z)$?

$$\mathbb{R}^{\cancel{3}2} \longrightarrow (\mathbb{R}^3)^* \times \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$x \longmapsto \langle A(x), q(x) \rangle \longmapsto [a(x)\ b(x)\ c(x)]\begin{bmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{bmatrix}$$

row vector

$A = [a(x)\ b(x)\ c(x)]$

$$DA_x = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$

$$D(Aq)_x(h) = [h_1 h_2 h_3]\begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}\begin{bmatrix} q^1 \\ q^2 \\ q^3 \end{bmatrix} + [a\ b\ c]\begin{bmatrix} q^1_x & q^1_y & q^1_z \\ q^2_x & q^2_y & q^2_z \\ q^3_x & q^3_y & q^3_z \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

Note how pattern of partials is different; $A \in (\mathbb{R}^3)^*$

② Now $A_{(p(x))}q(x)$

$$D((A \circ p)_{(x)} q(x))_x(h) = D(A \circ p)_x(h) \circledast q(x) + (A \circ p)_{(x)} \star Dq_x(h)$$

Only 1st term will be different:

$$DA_{p(x)} Dp_x(h)\, q(x)$$

$$[a(p^1(x,y), p^2(x,y))\ \ b(p^1(x,y), p^2(x,y))\ \ c(p^1(x,y), p^2(x,y))]$$

$$\sum a_i g_i \to \sum (a_i' g_i + a_i g_i')$$

1

$$H = [a(t)\ b(t)\ c(t)]\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \to [a'\ b'\ c']\begin{bmatrix} g \end{bmatrix} + [a\ b\ c]\begin{bmatrix} g_i' \\ g \\ g \end{bmatrix}$$

2

$$H = [a(x,y)\ b(x,y)\ c(x,y)]\begin{bmatrix} g(x) \\ g(x) \\ g(x) \end{bmatrix} \to \sum a_i g_i(x) = a(x) g_1(x) + b(x) g_2(x)$$

$$DH_x = \begin{bmatrix} \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} \end{bmatrix}$$

$$H_x = \frac{\partial a}{\partial x} g_1 + a \frac{\partial g_1}{\partial x} + \frac{\partial b}{\partial x} g_1 + b g_x' + c_x g' + c g_x'$$

$$H_y = a_y g' + a g_y' + b_y g' + b g_y' + c_y g' + c g_y'$$

$$[H_x\ H_y]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = [(a_x g_1 + a g_x' + b_x g^2 + b g_x^2 + c_x g^3 + c g_x^3)(a_y g' + a g_y' + b_y g^2 + b g_y^2 + c_y g^3 + c g_y^3)]$$

$$= [\cancel{(a_x + b_x + c_x) g_1 + (a+b+c) g_x'}\ \cancel{(a_y + b_y + c_y)}$$

$$= \left[[a_x\ b_x\ c_x]\begin{bmatrix} g^1 \\ g^2 \\ g^3 \end{bmatrix} h_1 + [a\ b\ c]\begin{bmatrix} g_x^1 \\ g_x^2 \\ g_x^3 \end{bmatrix} h_1 + [a_y\ b_y\ c_y]\begin{bmatrix} g \end{bmatrix} h_2 + [a\ b\ c]\begin{bmatrix} g_y \end{bmatrix} h_2 \right]$$

$$= \cancel{[a_x\ b_x\ c_x]} [h_1\ h_2]\begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} + [a\ b\ c]\begin{bmatrix} g_x^1 & g_y \\ g_x^2 & g^2 \\ g_x^3 & g^3 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

so now when we take it up a level

3

$$H = [a(p^1(x,y), p^2(x,y))\ \ b(p^1, p^2)\ \ c(p^1, p^2)]\begin{bmatrix} g(x) \\ g(x) \\ g(x) \end{bmatrix}$$

$D_1 a$ $\quad D_2 a$ $\quad D_1 b$

$$H_x = \left(\frac{\partial a}{\partial p^1} p_x^1 + \frac{\partial a}{\partial p^2} p_x^2\right) g_1 + a g_x^1 + \left(\frac{\partial b}{\partial p_1} p_x^1 + \frac{\partial b}{\partial p_2} p_x^2\right) g^2 + b g_x^2 + (c_{p_1} p_x^1 + c_{p_2} p_x^2) g^3 + c g_x^3$$

$$H_y = (a_1 p_y^1 + a_2 p_y^2) g^1 + a g_y^1 + (b_1 p_y^1 + b_2 p_y^2) g^2 + b g_y^2 + (c_1 p_y^1 + c_2 p_y^2) g^3 + c g_y^3$$

$$DH_x(h) = [H_x\ H_y]\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} =$$

$$h_1\Big((a_1 p^1_x + a_2 p^2_x) g^1 + (b_1 p^1_x + b_2 p^2_x) g^2 + (c_1 p^1_x + c_2 p^2_x) g^3 + a g^1_x + b g^2_x + c g^3_x \Big) +$$

$$h_2\Big((a_1 p^1_y + a_2 p^2_y) g^1 + (b_1 p^1_y + b_2 p^2_y) g^2 + (c_1 p^1_y + c_2 p^2_y) g^3 + a g^1_y + b g^2_y + c g^3_y \Big)$$

$$\begin{bmatrix} p^1_x & p^2_x \\ p^1_y & p^2_y \end{bmatrix}$$

$D_1 a$

$$\cancel{\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}}$$

Before

$$\begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$

$$\underbrace{[h_1\ h_2] \begin{bmatrix} p^1_x & \\ & p^2_y \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}}_{(a_1 p^1_x + a_2 p^1_y) g_1 + (b_1 p^1_x + b_2 p} \begin{bmatrix} g_1 \\ g^2 \\ g^3 \end{bmatrix} +$$

$$\begin{bmatrix} (a_1 p^1_x + a_2 p^2_x) & (b_1 p^1_x + b_2 p^2_x) & (c_1 p^1_x + c_2 p^2_x) \\ (a_1 p^1_y + a_2 p^2_y) & (b_1 p^1_y + b_2 p^2_y) & (c_1 p^1_y + c_2 p^2_y) \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

$$\Rightarrow [h_1\ h_2] \begin{bmatrix} p^1_x & p^2_x \\ p^1_y & p^2_y \end{bmatrix} \begin{bmatrix} D_1 a & D_1 b & D_1 c \\ D_2 a & D_2 b & D_2 c \end{bmatrix} \begin{bmatrix} g^1 \\ g^2 \\ g^3 \end{bmatrix} + [a\ b\ c] \begin{bmatrix} g^1_x & g^1_y \\ g^2_x & g^2_y \\ g^3_x & g^3_y \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

(4) Then the last calculation is going to be:

$$g(x) = G^h_{(x)} = Dg_x(h)$$

$$Dg_x(k) = D^2 g_x(h, k)$$

$$\begin{array}{ccccc} \mathbb{R}^2 & & \mathbb{R}^3 & & \mathbb{R} \\ X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \end{array}$$

$(x,y,z) \leftarrow x,y$

h, k

$$\begin{bmatrix} g^1 \\ g^2 \\ g^3 \end{bmatrix}$$

$$g = \begin{bmatrix} g^1 \\ g^2 \\ g^3 \end{bmatrix} \quad Dg = \begin{bmatrix} g^1_x & g^1_y & g^1_z \\ g^2_x & g^2_y & g^2_z \\ g^3_x & g^3_y & g^3_z \end{bmatrix}$$

Let's take the first component

$$= \begin{bmatrix} | & | & | \\ g_x & g_y & g_z \\ | & | & | \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

and the really last thing is $p(x) = g(x)$

$$D(Df_{g(x)} Dg(h))_x(k)$$

Recap again

$g : \mathbb{R}^2 \to \mathbb{R}^3$

we have to be sure that all these special cases have the right dimensions

A $Df_{g(x)} Dg_x(h)$

$\vec{g}$ is not $\vec{g}$

$\vec{g}$ is $Dg_x(h)$

Start from (2)

$$H = \begin{bmatrix} a(x,y) & b(x,y) & c(x,y) \end{bmatrix} \begin{bmatrix} g_1(x,y) \\ g_2 \\ g_3 \end{bmatrix}$$

$$DH_x(h) = \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \end{bmatrix} \begin{bmatrix} \\ g \\ \\ \end{bmatrix} + \begin{bmatrix} a & b & c \end{bmatrix} \underbrace{\begin{bmatrix} | & | \\ g_x & g_y \\ | & | \end{bmatrix}}_{DQ_x} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

(3) $H = \begin{bmatrix} a(\vec{p}(x,y)) & b(\vec{p}(x,y)) & c(\vec{p}(x,y)) \end{bmatrix} \begin{bmatrix} g \end{bmatrix}$

$$DH_x(k) = \underbrace{\begin{bmatrix} k_1 & k_2 \end{bmatrix}}_{k^T} \underbrace{\begin{bmatrix} p^1_x & p^2_x & p^3_x \\ p^1_y & p^2_y & p^3_y \end{bmatrix}}_{(DP_x)^T} \begin{bmatrix} D_1 a & D_1 b & D_1 c \\ D_2 a & D_2 b & D_2 c \\ D_3 a & D_3 b & D_3 c \end{bmatrix} \begin{bmatrix} g \end{bmatrix} + \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} | & | \\ g_x & g_y \\ | & | \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

(4) Now $g(x) = G^h(x) = Dg_x(h)$ and so $Dg_x(k) = D^2 g_x(h,k)$

$$\begin{bmatrix} | & | \\ g_x & g_y \\ | & | \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} g^1_x & g^1_y \\ g^2_x & g^2_y \\ g^3_x & g^3_y \end{bmatrix} \end{array} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

This is a vertical stack of case (2) 3 times and h not fcn of x;

$$DH_x(k) = k^T (DP_x)^T \begin{bmatrix} D_1 a & D_1 b & D_1 c \\ D_2 a & D_2 b & D_2 c \end{bmatrix} \begin{bmatrix} | & | \\ g_x & g_y \\ | & | \end{bmatrix} \begin{bmatrix} h \end{bmatrix} + \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} g^1_{xx} & g^1_{xy} \\ g^1_{yx} & g^1_{yy} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ \begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} g^2_{xx} & g^2_{xy} \\ g^2_{yx} & g^2_{yy} \end{bmatrix} \begin{bmatrix} h \end{bmatrix} \\ \begin{bmatrix} k \end{bmatrix} \begin{bmatrix} g^3_{xx} & g^3_{xy} \\ g^3_{yx} & g^3_{yy} \end{bmatrix} \begin{bmatrix} h \end{bmatrix} \end{bmatrix}$$

(5) And the last thing for

$D\left(Df_{g(x)} Dg_x(h) \right)_x (k)$

is for $p(x) = g(x)$

thus

$$DH_x(k) = k^T (Dg_x)^T D^2 f_{g(x)} Dg_x(h) + Df_{g(x)} \begin{bmatrix} k^T D^2 g^{(1)}_x(h) \\ k^T D^2 g^{(2)}_x(h) \\ k^T D^2 g^{(3)}_x(h) \end{bmatrix}$$

$$= (Dg_x(k))^T D^2 f_{g(x)} Dg_x(h) + Df_{g(x)} D^2 g(h,k)$$