Thm $f: E\times E \longrightarrow F$, $(x_1,x_2) \longmapsto b(x_1,x_2)$ bilinear (linear in both positions) $\Rightarrow Df_x(h) = b(x_1,h_2) + b(h_1,x_2)$ [b not nec symm]

Pf. $f(x+h) - f(x) = b(x_1+h_1, x_2+h_2) - b(x_1,x_2)$

$$= \cancel{b(x_1,x_2)} + \underbrace{b(x_1,h_2) + b(h_1,x_2)}_{Df_x(h)} + b(h_1,h_2) - \cancel{b(x_1,x_2)}$$

Show $\dfrac{\|b(h_1,h_2)\|_F}{\|h\|_{E_1\times E_2}} \xrightarrow[h\to 0]{} 0$

$$\|b(h_1,h_2)\| \le \|b\|_{op} \|h_1\|_{E_1} \|h_2\|_{E_2}$$
$$\le \|b\| \left[|h_1||h_2| + |h_1|^2 + |h_1||h_2| + |h_2|^2\right]$$

$|h|_{E\times E} = |h|_{E_1} + |h|_{E_2}$

From the ineq $p>0$, $q>0$
$$pq \le (p+q)^2 = p^2 + 2pq + q^2$$
$$= \|b\|_{op} \left[|h_1|_{E_1} + |h_2|_{E_2}\right]^2$$
$$= \|b\|_{op} \|h\|^2_{E_1\times E_2} \longrightarrow 0$$

□

The most common case would be
$f: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$
$(x_1,x_2) \longmapsto x_1^T A x_2$ Then $Df_x(h) = x_1^T A h_1 + h_2^T A x_2$ [cf. M&T ch 4.2 writeup]

If we made this quadratic $f: \mathbb{R}^n \to \mathbb{R}$
$x \longmapsto x^T A x$ we'd get $Df_x(h) = x^T A h + \underbrace{h^T A x}_{= x^T A^T h}$ ← Real number, so it equals transpose

$\Rightarrow Df_x(h) = x^T(A + A^T)h$

And if b is symm: $b(x,y) = b(y,x) \Rightarrow A = A^T$

$Df_x(h) = 2x^T A h \qquad Df_x(\cdot) = 2x^T A$.

Thm Chain Rule $D(g\circ f)_x(\cdot) = Dg_{f(x)} \cdot Df_x(\cdot)$

See my writeup of M&T ch 2.4 for special cases for $\mathbb{R}^n$

$X \supseteq \mathcal{U} \xrightarrow{f} Y \xrightarrow{g} Z$

Cheney ch 4 p.5
POMA p. 214

Pf. Fix $x \in \mathcal{U}$. Let $y = f(x)$ and $A := Df_x$ $\quad B := Dg_y \quad F(x) := (g\circ f)(x)$

Want: $\displaystyle\lim_{h\to 0} \frac{|F(x+h) - F(x) - BAh|}{|h|} = 0$

Since f dif'b:

(*) $f(x+h) - f(x) - Ah = \sigma_1(h)$ where $\dfrac{|\sigma_1(h)|}{|h|} \to 0$ as $h \to 0$

(**) $g(y+k) - g(y) - Bk = \sigma_2(k)$ $\quad |\sigma_2(k)|/|k| \to 0$ as $k \to 0$

$$F(x+h) = g(f(x+h)) \text{ by dof}$$
$$= g(\underbrace{f(x)}_{y} + \underbrace{Ah + \sigma_1(h)}_{k}) \text{ by (*)}$$

← define this to be the vector k.

$$= g(y+k)$$
$$= g(y) + Bk + \sigma_2(k) \text{ by (**)}$$
$$= g(f(x)) + B(Ah + \sigma_1(h)) + \sigma_2(k) \text{ by def of } k$$
$$= F(x) + BAh + B\sigma_1(h) + \sigma_2(k)$$

$$\Rightarrow F(x+h) - F(x) - BAh = B\sigma_1(h) + \sigma_2(k)$$

Cont'd →

cont'd So we need to show: $\frac{1}{|h|}|F(x+h)-F(x)-BAh| = \frac{1}{|h|}|B\theta_1(h)+\theta_2(k)| \longrightarrow 0$ as $h\to 0$

$$\frac{1}{|h|}|F(x+h)-F(x)-BAh| \le \frac{1}{|h|}|B\theta_1(h)| + \frac{1}{|h|}|\theta_2(k)|$$

$$\le \|B\|_{op}\frac{|\theta_1(h)|}{|h|} + \frac{|\theta_2(k)|}{|h|}$$

$k = Ah + \theta_1(h)$

$|k| \le |A||h| + |\theta_1(h)|$

$$= \|B\|_{op}\frac{|\theta_1|}{|h|} + \frac{|\theta_2(k)|}{|k|}\frac{|k|}{|h|} \le |B|\frac{|\theta_1|}{|h|} + \frac{|\theta_2|}{|k|}\left[|A| + \frac{|\theta_1(h)|}{|h|}\right]$$

$$\lim_{h\to 0}\frac{|F(x+h)-F(x)-BAh|}{|h|} \le |B|\underbrace{\frac{|\theta_1(h)|}{|h|}}_{0} + \underbrace{\frac{|\theta_2(k)|}{|k|}}_{0}\left[|A| + \underbrace{\frac{|\theta_1(h)|}{|h|}}_{0}\right] = 0$$

$|h|\to 0 \Rightarrow |k|\to 0$

$\boxed{\text{QED}}$

Tangent Map

For $f: \overset{E}{\mathcal{U}} \to F$ we have $Df: \mathcal{U} \to \mathcal{L}(E\to F)$, $u \mapsto Df_u$

Define $Tf: \mathcal{U}\times E \to F\times F$, $(u,e) \mapsto (f(u), Df_u(e))$

The idea is that this map behaves better under composition:

$E \xrightarrow{f} F \xrightarrow{g} G$ then $g\circ f: \mathcal{U}\to G$

$$T(g\circ f)_u(e) = Tg(f(u), Df_u(e)) = \left(g(f(u)), Dg_{f(u)}(Df_u(e))\right)$$

really (u,e)

$$= \left((g\circ f)(u),\ D(g\circ f)_u(e)\right)$$

They say this is the advantage: $T(g\circ f) = Tg\circ Tf$ but $D(g\circ f) = Dg_{f(\cdot)}\cdot Df_{(\cdot)}$

('covariant functor')

ch 1.2.7 Mapping INTO a Direct sum

$f: E \to \underbrace{F_1\oplus F_2\oplus\cdots\oplus F_n}_{F}$

Component fcns

Define projection $\pi: F \twoheadrightarrow F_i$, $\langle x^1,\dots,x^n\rangle \mapsto x^i$

Injection $\imath: F_i \hookrightarrow F$, $x^i \mapsto \langle 0,\dots,0,x^i,0\cdots 0\rangle$

For $f: E\to F$, $x \mapsto \langle f^{1}(x),\dots,f^{n}(x)\rangle$

$f^i = \pi_i\circ f$

$f = \sum_{i=1}^n \imath_i\circ f^i(x)$ using the fact that F is a vector sp. So we can add.

Thm 1.1 f dif'b at $a\in\mathcal{U} \iff f^{1},\dots,f^{n}$ all dif'b at a

$$Df_a = \langle Df_a^1,\dots,Df_a^n\rangle$$

Pf. apply chain rule to $f = \sum_{i=1}^n \imath_i\circ f^i(x)$ □

COR $f:\mathbb{R}^n\to\mathbb{R}^m$ if dif'b at $a \Rightarrow$ i^{th} row of Df_a is Df_a^i

More on maps $\mathbb{R}^n\to\mathbb{R}^m$ later.

ch 1.2.8 Leibniz formula [General bilinear-product product rule]

How do we differentiate something like $\vec{f}(x)\cdot\vec{g}(x)$ or $\vec{f}(x)\times\vec{g}(x)$?

These are both examples of a bilinear map on 2 vector valued fcns.

$f * g = b(f,g)$

To be more general, lets define $\boxed{fg}: E \xrightarrow{\lambda} F_1 \times F_2 \xrightarrow{b} G$

$x \longmapsto \langle f(x), g(x)\rangle \longmapsto b(f(x), g(x))$

Thm $D\boxed{fg}_x(h) = b(Df_x(h), g(x)) + b(f(x), Dg_x(h))$

See also ch 4

If $f, g \in C^1$, then this is obviously cont, so $\boxed{fg} \in C^1$ also cf ch 4.2 writeup

Pf. By chain rule, $D\boxed{fg}_x(\cdot) = Db_{\lambda(x)} D\lambda_x(\cdot)$

First compute $D\lambda_x$ $\quad \lambda(x+h) - \lambda(x) = (f(x+h), g(x+h)) - (f(x), g(x))$

$= (f(x+h) - f(x),\ g(x+h) - g(x))$

Thus we take $D\lambda_x(h) = (Df_x(h), Dg_x(h))$

Then $D\boxed{fg}_x = Db_{\lambda(x)} D\lambda_x$

$= Db_{(f,g)}(Df_x(h), Dg_x(h))$ ← from sheet 5 $Db_{x_1,x_2}(h_1,h_2) = b(x_1,h_2) + b(h_1,x_2)$

$= b(Df_x(h), g(x)) + b(f(x), Dg_x(h))$

□

Let's consider some applications:

- ordinary mult $f, g: \mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad f(x)g(x)$

  $b(f,g) = fg$ $\quad$ Product rule $D(fg)_x(h) = b(Df_x(h), g(x)) + b(f(x), Dg_x(h))$

  $= Df_x(h)g(x) + f(x)Dg_x(h)$

  [If $n=1$ we get regular product rule: $[f'(x)g(x) + f(x)g'(x)]h$ take $h=1$

- Dot product $\vec{f}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $\vec{g}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

  $b(\vec{f},\vec{g}) = \vec{f}\cdot\vec{g}$ $\quad D(\vec{f}\cdot\vec{g})_x(h) = \overrightarrow{Df_x(h)}\cdot\vec{g}(x) + \vec{f}(x)\cdot\overrightarrow{Dg_x(h)}$

- Cross-prod $b(\vec{f},\vec{g}) = \vec{f}\times\vec{g}$ $\quad D(\vec{f}\times\vec{g})_x(h) = \overrightarrow{Df_x(h)}\times\vec{g} + f(x)\times\overrightarrow{Dg_x(h)}$

- Scalar mult of vector $b(\alpha,\vec{f}) = \alpha(x)\vec{f}(x)$ $\quad D(\alpha f)_x(h) = D\alpha_x(h)\vec{f}(x) + \alpha(x)\overrightarrow{Df_x(h)}$

  M&T ch 2.4 p.128

- Matrix multiplication $A\vec{g} = [a\ b\ c]\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$

  $\mathbb{R} \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^1$

  $t \longmapsto \langle A(t), g(t)\rangle \longmapsto A(t)\vec{g}(t) = b(A,g)$

By the rule

$D\boxed{Ag}_t(h) = DA_t(h)\vec{g}(t) + A(t)Dg_t(h)$

$= h[\dot{a}\ \dot{b}\ \dot{c}]\begin{bmatrix} g \end{bmatrix} + [a\ b\ c]\begin{bmatrix} \dot{g}_1 \\ \dot{g}_2 \\ \dot{g}_3 \end{bmatrix}h$

Here we don't really 'need' h for the pieces to fit together

Now consider $A(x,y,z)\vec{g}(x,y,z)$ $\quad$ Now $h \in \mathbb{R}^3$

$\mathbb{R}^3 \longrightarrow \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$

$x \longmapsto \langle A(x), g(x)\rangle \longmapsto A(x)\vec{g}(x)$

$D\boxed{Ag}_x(h) = DA_x(\vec{h})\vec{g}(x) + A(x)Dg_x(h)$

$A = [a\ b\ c]$ $\quad DA_x = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$

$\Rightarrow [h_1\ h_2\ h_3]\begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} + [a\ b\ c]\begin{bmatrix} g^1_x & g^1_y & g^1_z \\ g^2_x & g^2_y & g^2_z \\ g^3_x & g^3_y & g^3_z \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$

Note the pattern of partial derivs because A is a row vector but g is a col vector.

Creating these matrices is actually discussed further on

Ch 1.2.9 Maps defined ON a direct sum

$$f: E_1 \oplus E_2 \oplus \cdots \oplus E_n \longrightarrow F \quad \text{nls}$$
$$\langle x_1, \dots, x_n \rangle \longmapsto f(x)$$

big idea: $\boxed{\text{Partial Derivs}}$

Define inclusion map $\mathbb{I}_j^a : x_j \hookrightarrow \langle a_1, \dots x_j \dots a_n \rangle$

Define $f^{(j)} = f \circ \mathbb{I}_j$ $\qquad a = \langle a_1 \dots a_n \rangle$

Avez's discussion is hard to follow here. He wants basically to define

$$g_a : E_j \longrightarrow F$$
$$x_j \longmapsto f(a_1, \dots, a_{j-1}, x_j, a_{j+1} \dots a_n)$$

Then we can compute $D(g_a)_{a_j}$ and call it partial deriv $D_j f(a)$

I am going to shift to the important special case $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Rudin POMA p. 215

Let $\mathbb{R}^n$ have std O.N. basis $\{e_1 \dots e_n\}$ and $\mathbb{R}^m$ has the same, but notated $\{u_1, \dots u_m\}$

$$\text{Then } f(x) = \sum_{i=1}^m f^i(x)\, \hat{u}_i = \{u_1 \dots u_m\} \begin{bmatrix} f^1(x) \\ \vdots \\ f^m(x) \end{bmatrix} \text{ by std convention}$$

Define partial $\dfrac{\partial f^i}{\partial x_j}(x) = D_j f^i(x) = \lim\limits_{t \to 0} \dfrac{f^i(x + t\hat{e}_j) - f(x)}{t}$

Thm f difb at x $\Rightarrow$ all partials $\dfrac{\partial f^i}{\partial x_j}$ exist at x (not nec cont) and

$$Df_x = \left[\frac{\partial f^i}{\partial x_j} \right] \text{ matrix}$$

POMA p. 215

Pf. f difb $\Rightarrow \exists\, A_x : \mathbb{R}^n \to \mathbb{R}^m \ni f(x+h) - f(x) = Ah + o(h)$ $\qquad$ A is unique

Fix j $1 \le j \le n$ so $\hat{e}_j$ is a fixed basis vector

Let $h = t\hat{e}_j$ $\qquad \lim\limits_{t \to 0} \dfrac{\vec{f}(x + te_j) - \vec{f}(x)}{t} = \dfrac{1}{\cancel{t}} A(\cancel{t}e_j) + \lim\limits_{t \to 0} \cancel{\dfrac{o(te_j)}{t}}^{\,0}$

$$\lim_{t \to 0} \begin{bmatrix} \dfrac{f^{(1)}(x+te_j) - f^1(x)}{t} \\ \vdots \\ \dfrac{f^{(m)}(x+te_j) - f^m(x)}{t} \end{bmatrix} = Ae_j = \begin{bmatrix} a_j^{(1)} \\ \vdots \\ a_j^{(m)} \end{bmatrix}$$

$$\| \qquad\qquad \|$$

$$\begin{bmatrix} \dfrac{\partial f^1}{\partial x_j}(x) \\ \vdots \\ \dfrac{\partial f^m}{\partial x_j}(x) \end{bmatrix}$$

This is the j^{th} column
You get all cols by going thru all basis vectors $\hat{e}$ $\square$

See my writeup of M&T ch 2.3 where I put the writeup that all partials <u>continuous</u> implies Df_x exists and is, in fact, C^1. [It was from Rudin POMA ch 9 Thm 9.21]

I did a more detailed discussion of using the chain rule for maps $\mathbb{R}^n \to \mathbb{R}^m$ in my writeup of M&T ch 2.4.

p.9 Thm 1.4 COV

$$\begin{tikzcd} & \mathbb{R} & \\ \mathbb{R}^n \arrow[ur, "F"] \arrow[rr, "g"] & & \mathbb{R}^m \arrow[ul, "f"'] \end{tikzcd}$$

a

$$\left.\begin{array}{l} g: \mathbb{R}^n \to \mathbb{R}^m \text{ difb at } a \\ f: \mathbb{R}^m \to \mathbb{R} \text{ difb at } g(a) \\ F = f\circ g \end{array}\right\} \Rightarrow \begin{array}{l} DF_a = Df_{g(a)} \cdot Dg_a \\ [D_1F \cdots D_nF] = [D_1f \cdots D_mf]\begin{bmatrix} g^1_1 & \cdots & g^1_n \\ g^m_1 & \cdots & g^m_n \end{bmatrix} \end{array}$$

This is also Spivak COM p.32

Let's consider an example: $F: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\langle x,y\rangle \longmapsto f(\underbrace{h(x)}_{g^1(x,y)}, \underbrace{s(y)}_{g^2(x,y)})$

Then the pattern is $D_1F = \dfrac{\partial F}{\partial x} = \dfrac{\partial f}{\partial g^1}\dfrac{\partial g^1}{\partial x} + \cancel{\dfrac{\partial f}{\partial g^1}\dfrac{\partial g^1}{\partial y}} + \dfrac{\partial f}{\partial g^2}\dfrac{\partial g^2}{\partial x} + \cancel{\dfrac{\partial f}{\partial g^2}\dfrac{\partial g^2}{\partial y}}$

$= \dfrac{\partial f}{\partial h} h'(x) + \dfrac{\partial f}{\partial s}\underbrace{s'(x)}_{0}$

$D_2F = \dfrac{\partial F}{\partial y} = \dfrac{\partial f}{\partial g^1}\underbrace{\cancel{\dfrac{\partial g^1}{\partial y}}}^{0} + \dfrac{\partial f}{\partial g^2}\dfrac{\partial g^2}{\partial y} = \dfrac{\partial f}{\partial s} s'(y)$

But if we added another layer $x = x(t)$, $y = y(t)$ $\quad H(t) := f(g^1(x(t),y(t)), g^2(x(t),y(t)))$

Then $\dfrac{dH}{dt} = Dh_t = \dfrac{\partial f}{\partial g^1}\dfrac{\partial g^1}{\partial x}\dot{x} + \dfrac{\partial f}{\partial g^1}\dfrac{\partial g^1}{\partial y}\dot{y} + \dfrac{\partial f}{\partial g^2}\dfrac{\partial g^2}{\partial x}\dot{x} + \dfrac{\partial f}{\partial g^2}\dfrac{\partial g^2}{\partial y}\dot{y}$

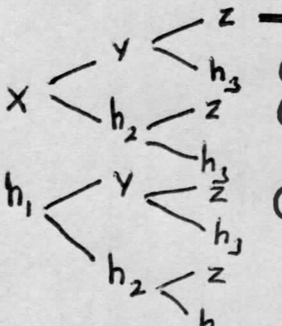
$= \dfrac{\partial f}{\partial g^1}\left(\dfrac{\partial g^1}{\partial x}\dot{x} + \dfrac{\partial g^1}{\partial y}\dot{y}\right) + \dfrac{\partial f}{\partial g^2}\left(\dfrac{\partial g^2}{\partial x}\dot{x} + \dfrac{\partial g^2}{\partial y}\dot{y}\right)$

▷ Now let's consider differentiating a determinant:

First recall sheet ⑤ for bilinear maps $b(x,y)$: $Df_{xy}(h_1 h_2) = b(h_1,y) + b(x,h_2)$

Now, what is the deriv for a 3-linear map? Follow the same pattern as sheet ⑤

$q(x+h_1, y+h_2, z+h_3) = q(x, y+h_2, z+h_3) + q(h_1, y+h_2, z+h_3)$ there are going to be a lot of terms, let's show it in a diagram



(x,y,z)
(x,y,h_3) $O(1)$
(x,h_2,z)
(x,h_2,h_3) $O(2)$
(h_1,y,z)
(h_1,y,h_3) $O(2)$
(h_1,h_2,z) $O(2)$
(h_1,h_2,h_3) $O(3)$

we only want the $O(1)$ terms in h:

$$\boxed{Dq_{xyz}(h_1 h_2 h_3) = q(h_1,y,z) + q(x,h_2,z) + q(x,y,h_3)}$$

The det is a multi-linear map, So we know what D(det) is from this pattern.

Spivak COM p.24 gives this as exercises

Cont'd →

$$\det : \overbrace{\mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n}^{n} \longrightarrow \mathbb{R}$$

Lets just show the 3x3 case:

$$g(x,y,z) = \det \begin{bmatrix} - x - \\ - y - \\ - z - \end{bmatrix} \qquad D(\det)_{xyz}(h_1 h_2 h_3) = \det \begin{bmatrix} - h_1 - \\ - y - \\ - z - \end{bmatrix} + \det \begin{bmatrix} - x - \\ - h_2 - \\ - z - \end{bmatrix} + \det \begin{bmatrix} - x - \\ - y - \\ - h_3 - \end{bmatrix}$$

▷ Now lets consider the case where each element is a fcn of t [cf M&T Ch 7.■4 Trasport Thm]

Lets show this 2 ways

① For a smooth map $f: \mathbb{R}^N \to \mathbb{R}$ let $h(t) = f(\sigma(t))$
then $Dh_t(1) = Df_{\sigma(t)}(\dot{\sigma}(t))$ because $Df_{\sigma(t)} D\sigma_t(1)$

We could write a nxn matrix as a long vector in $\mathbb{R}^{n^2}$ and regard det as a map $f: \mathbb{R}^{n^2} \to \mathbb{R}$ satisfying certain multi-linearity properties.

Lets take the 2x2 case and bilinearity: $f: \mathbb{R}^{2^2} \longrightarrow \mathbb{R}$

$$\begin{bmatrix} \vec{x} \\ \hline \vec{y} \end{bmatrix} \longmapsto b(x,y)$$

$$\text{Then } Df_{\substack{x\\y}} \begin{bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \end{bmatrix} = \underbrace{\big(b(\cdot, y) + b(x, \cdot)\big)}_{\substack{\text{This whole thing is}\\ \text{the linear map applied to}\\ \text{a long vector}}} \begin{bmatrix} \dot{\sigma}_1 \\ \cdots \\ \dot{\sigma}_2 \end{bmatrix} = b(\dot{\sigma}_1, y) + b(x, \dot{\sigma}_2)$$

$$\boxed{= \det \begin{bmatrix} - \dot{\sigma}_1 - \\ - y - \end{bmatrix} + \det \begin{bmatrix} - x - \\ - \dot{\sigma}_2 - \end{bmatrix}}$$

Likewise for the 3x3 case and nxn.

② Now let's show another nice arg from ~~math/stack/exchange~~:

Take 3x3 case.

We want $\displaystyle\lim_{h\to 0} \frac{g(a(t+h), b(t+h), c(t+h)) - g(a(t), b(t), c(t))}{h}$

Just look at numerator, add and subtract terms. For convenience, abbreviate $a(t+h)$ as a_{th}, $a(t) = a$

$$g(a_{th}, b_{th}, c_{th}) - g(a,b,c) = \overbrace{g(a_{th}\ b_{th}\ c_{th}) - g(a, b_{th}, c_{th})} + \overbrace{g(a, b_{th}, c_{th}) - g(a,b,c_{th})} + \overbrace{g(a,b,c_{th}) - g(a,b,c)}$$

linearity

$$= g([a_{th} - a], b_{th}\ c_{th}) + g(a, [b_{th} - b], c_{th}) + g(a, b, [c_{th} - c])$$

Now mult both sides by $\frac{1}{h}$

$$\frac{g(a_{th}\ b_{th}\ c_{th}) - g(a,b,c)}{h} = g\left(\left[\frac{a_{th} - a}{h}\right], b_{th}\ c_{th}\right) + g\left(a, \left[\frac{b_{th} - b}{h}\right], c_{th}\right) + g\left(a, b, \left[\frac{c_{th} - c}{h}\right]\right)$$

Now take $\lim_{h\to 0}$ both sides and lim goes inside on RHS because g is cont. $b_{th} \longrightarrow b_t = b$

$$\frac{d}{dt} g(a(t), b(t), c(t)) = g(\dot{a}, b, c) + g(a, \dot{b}, c) + g(a, b, \dot{c})$$

$$\det \begin{bmatrix} - \dot{a} - \\ - b - \\ - c - \end{bmatrix} + \det \begin{bmatrix} - a - \\ - \dot{b} - \\ - c - \end{bmatrix} + \det \begin{bmatrix} - a - \\ - b - \\ - \dot{c} - \end{bmatrix}$$

QED

There is also Jacobi's formula

$$\frac{d}{dt} A(t) = \mathrm{tr}(\mathrm{adj}(A(t))\, \dot{A}(t)) = \det(A(t)) \cdot \mathrm{tr}(A^{-1}(t) \cdot \dot{A}(t))$$

adj = classical adjoint

Special case: $\frac{\partial}{\partial A_{ij}} \det(A) = \mathrm{adj}(A)_{ji}$