Avez DC Ch 1

Also cheney ch 4-1
POMA p. 212
Spivak COM p. 16

Summary and Extensions

See also my Marsden & Tromba VC Ch 2 write up where I covered some of this material, and extras, with emphasis on $\mathbb{R}^n$

3/26/2022

Let X, Y be n.l.s. (normed linear space). open set $\mathcal{U} \subseteq X$

$$f : \overset{X}{\mathcal{U}} \longrightarrow Y$$

Def f is dif'b at $x \in \mathcal{U}$ if: $\exists$ a Bdd Linear map $A_x : X \to Y$ satisfying

$\exists$ const $M \ni \|Ax\| \le M\|x\| \ \forall x$

Frechet Deriv (later we get to Gateaux) See sheet 1a

$$\lim_{\|h\| \to 0} \left[\frac{\|f(x+h) - f(x) - A_x h\|_Y}{\|h\|_X} \right] = 0$$

equivalently: $$\lim_{\|h\| \to 0} \left\| \frac{1}{\|h\|}[f(x+h) - f(x)] - A_x \hat{h} \right\|_Y = 0$$

We denote A_x by Df_x

Thm The linear map A_x is unique

Pf. Suppose there were 2 A_x and B_x

Then Given $\varepsilon > 0$, $\|h\| < \delta_A \Rightarrow \|\underbrace{f(x+h) - f(x)}_{F} - A_x h\| < \varepsilon\|h\|$

$\|h\| < \delta_B \Rightarrow \|f(x+h) - f(x) - B_x h\| < \varepsilon\|h\|$

Take $\delta := \min\{\delta_A, \delta_B\}$

$$\|F - Ah - (F - Bh)\| \overset{\text{Triang}}{\le} \|F - Ah\| + \|F - Bh\| < 2\varepsilon\|h\|$$

$$\|Bh - Ah\| \le 2\varepsilon\|h\|$$

$$\|(B-A)\hat{h}\| \le 2\varepsilon$$

$$\sup_{\text{all } \|\hat{h}\|=1} \|(B-A)\hat{h}\| \le 2\varepsilon \implies \|B-A\|_{op} \le 2\varepsilon$$

but since ε is arb small and there is no h anymore, we must have $\|B-A\|_{op} = 0$

$\Rightarrow B = A$ $\square$

Thm f dif'b at x $\Longrightarrow$ f cont at x

C^{0+d}

In fact, f is Lipschitz Cont: $\|f(x) - f(y)\| \le M\|x-y\|$ at fixed x

Pf. We have that, given any $\varepsilon > 0$, $\exists \delta > 0 \ni \|h\| < \delta \Rightarrow \|f(x+h) - f(x) - A_x h\| < \varepsilon\|h\|$

apply rev triang ineq: $|a| - |b| \le |a - b|$

$$\|f(x+h) - f(x)\| - \|A_x h\| \le \|f(x+h) - f(x) - A_x h\| < \varepsilon\|h\|$$

$$\|f(x+h) - f(x)\| < \varepsilon\|h\| + \|A_x h\|$$

$< [\varepsilon + \|A_x\|_{op}]\|h\|$ since A_x Bdd Linear

Thus $\|f(x+h) - f(x)\| < \overbrace{[1 + \|A\|]}^{M}\|h\|$

$< [\varepsilon + \|A_x\|_{op}]\delta$ so, by shrinking δ if nec, take $\delta < \dfrac{\varepsilon}{\varepsilon + \|A\|}$

$< \varepsilon$

$\square$

Avez DC ch 1 — **Frechet vs. Gateaux** — $f: X \to Y$ think $X = C[0,1] = Y$

$\frac{\partial f}{\partial \hat{h}}(x)$ Gateaux deriv $\delta f(x;h) = \delta f_x(h) := \lim_{t\to 0} \frac{1}{t}\left[f(x+th) - f(x)\right]$

Given $\varepsilon > 0$, $\exists$ a pt $Q_x(h) \in Y$ (this depends on the "direction" $\hat{h}$ - unit vector)
$\exists\ \delta_h > 0$ (depends on h)
such that $|t| < \delta_h \Rightarrow \left\| \frac{1}{t}[f(x+th) - f(x)] - Q_x(h) \right\| < \varepsilon$ — we symbolize this as $\delta f_x(h)$

Rosenthal, in his class notes, introduces the notion of a uniform limit which means δ in indep of the "direction" $\hat{h}$ — that means Given $\varepsilon>0$, the same δ works for any $\hat{h}$ ($\|h\|$ matters)

Thm Gateaux + Uniformity & Linearity conds $\Leftrightarrow$ Frechet

This means Given $\varepsilon>0$ $\exists\ \delta$ indep of the 'direction' $\hat{h}$ ($\|\hat{h}\|=1$), whatever 'direction' means in $C[0,1]$

Conds here based on Rosenthal class 2/21/91

$$\left.\begin{array}{ll} \delta f_x(h) = \lim_{t\to 0} \frac{1}{t}[f(x+th) - f(x)] & \text{this limit exists uniformly } \forall h \in S(0,1) \\ \delta f_x(\alpha \cdot h + b) = \alpha\, \delta f_x(h) + \delta f_x(b) & \text{Linear (Bdd Linear) in } h \text{ position} \end{array}\right\} \Longleftrightarrow$$

Frechet deriv $Df_x(\cdot)$ exists at x [and $\delta f_x(h) = Df_x(h)$]

Pf ($\Leftarrow$) By def of Frechet Deriv: Given $\varepsilon>0$ $\exists$ Bdd Linear map A_x (indep of h) and $\exists\ \delta>0$ (indep of h) $\ni$

$$(\star)\quad \|h\|_X < \delta \Rightarrow \left\| \frac{1}{\|h\|_X}[f(x+h) - f(x)] - \frac{A_x(h)}{\|h\|_X} \right\|_Y < \varepsilon$$

Restrict the set of h's we consider to $h = tu$ for some fixed u (not nec unit $\hat{u}$)

Then $(\star)$ becomes $\|tu\| < \delta$ (indep of $\hat{u}$) $\Rightarrow \left\| \frac{1}{\|tu\|}[f(x+tu) - f(x)] - \frac{A(tu)}{\|tu\|} \right\| < \varepsilon$

that is $|t| < \frac{\delta}{\|u\|} \Rightarrow \left\| \frac{1}{|t|}[f(x+tu) - f(x)] - \frac{t\,A(u)}{|t|} \right\| < \varepsilon \|u\|$ — fixed

This only depends on magnitude $\|u\|$, not 'dir' and $\|u\|$ here is a fixed number

- If t were pos, we could drop abs val and get
$\left\| \frac{1}{t}[f(x+tu) - f(x)] - A_x(u) \right\| < \varepsilon \|u\|$ — indep of t
- If t were neg, we could replace $|t|$ by $-t$ and get
$\left\| \frac{1}{-t}[f(x+tu) - f(x)] - \frac{t\,A_x(u)}{-t} \right\| = \left\| (-1)\left(\frac{1}{t}[f(x+tu) - f(x)] - A_x(u)\right) \right\| < \varepsilon \|u\|$

Same result in both cases.

Avez p.1-2 just states $Df_a(k) = \frac{d}{dt} f(a+tk)\Big|_{t=0}$
This is misleading! Let $\sigma(t) = a + tk$
$$\frac{d}{dt} f(\sigma(t))\Big|_{t=0} = \lim_{t\to 0} \frac{f(\sigma(t)) - f(\sigma(0))}{t} = \lim_{t\to 0} \frac{f(a+tk) - f(a)}{t} = \delta f_a(k)$$

Now since ε is arb small, this is saying

$$\lim_{t\to 0} \frac{1}{t}[f(x+tu) - f(x)] = A_x(u)$$
$$\delta f_x(u) = A_x(u)$$

and since RHS is linear in u so is LHS

$$\left\{\begin{array}{l} \text{Plug in } u = \alpha e + b \\ \delta f_x(\alpha e + b) = A(\alpha e + b) \\ \qquad = \alpha \underbrace{A(e)}_{=\delta f_x(e)} + \underbrace{A(b)}_{=\delta f_x(b)} \end{array}\right\}$$

Observe that once we get the chain rule, this is trivial
$$\frac{d}{dt} f(\sigma(t))\Big|_{t=0} = Df_{\sigma(t)}(\sigma'(t))\Big|_{t=0} = Df_a(k)$$

This shows Frechet exists at x $\Rightarrow$ Gateaux exists at x
$\delta f_x(\cdot)$ is Bdd Linear since $A_x(\cdot)$ is
Gateaux has uniform limit (does not depend on direction $\hat{u}$). $\square$

ASIDE 'Easy Fact' Rosenthal 2/19/91 using his notation: $\frac{\partial f}{\partial(\lambda \vec{e})}(x) = \lambda \frac{\partial f}{\partial \vec{e}}(x)$

Pf $\lim_{t\to 0} \frac{f(x+t\lambda e) - f(x)}{t} = \lim_{t\to 0} \lambda \left[\frac{f(x+t\lambda e) - f(x)}{\lambda t}\right] \overset{\text{cov } s := \lambda t,\ t\to 0 \Rightarrow s \to 0}{=} \lambda \lim_{s\to 0}\left[\frac{f(x+s\vec{e}) - f(x)}{s}\right] = \lambda \frac{\partial f}{\partial e}(x)$ $\square$

Cont'd →

Cont'd $\underline{\text{renaming}}$ "X" = Y, "Y" = $\mathbb{R}$ "f" = J "x" = y_0

To show Gateaux + uniformity $\Longrightarrow$ Frechet, I will cut corners here and give my work thru of the proof in Troutman VCWEC p.119-120 Thm 5.9, where $f: X \to Y = \mathbb{R}$ (so X can still be C[0,1], but Y is restricted to be $\mathbb{R}$, and we rename f as "J"). Troutman also changes x to y_0. He renames "X" as "Y"

$\overset{\circ}{Y}$

$$\left.\begin{array}{l} J: B(y_0, r) \longrightarrow \mathbb{R} \\ \delta J_y(v) \text{ exists } \forall y \in B(y_0,r) \text{ and all directions } v \in Y \\ \text{(a) } \delta J_{y_0}(v) \text{ is linear and Cont in v position (Bdd Linear)} \\ \text{(b) } |\delta J_y(u) - \delta J_{y_0}(u)| \to 0 \ \underline{\text{uniformly}} \ \forall \hat{u} \in S(0,1) \end{array}\right\} \Rightarrow \text{ J is frechet difb at } y_0$$

This means: Given $\varepsilon > 0$, $\exists\, \delta > 0$ (indep of u; same δ works for any $\hat{u}$) $\ni \|y - y_0\| < \delta \Rightarrow |\delta J_y(u) - \delta J_{y_0}(u)| < \varepsilon$

This means Given $\varepsilon > 0$, $\left.\begin{array}{l}\exists \text{ number } Q_y(u) \\ \exists\, \delta_u > 0 \text{ (depends on } \hat{u}) \end{array}\right\} \ni |tu| < \delta_u \Rightarrow \left|\frac{J(y_0+tu) - J(y_0)}{t} - Q_{y_0}(u)\right| < \varepsilon$

we must show Q equals this

$\delta J_{y_0}(u)$ is how this number is symbolized

what we $\underline{\text{want}}$ is: Given $\varepsilon > 0$ $\exists\, \delta > 0$ (indep of dir of $\hat{h}$) $\exists$ Bdd Linear map A_{y_0} (indep of h) $\ni \|h\| < \delta \Rightarrow \left|\frac{J(y_0+h) - J(y_0) - A_{y_0}(h)}{\|h\|}\right| < \varepsilon$

$\underline{\text{Pf}}$ **Step 1** we define $\underline{A_{y_0}(u)}$ for $\underline{\text{each } u}$ to be $Q_{y_0}(u)$ $[= \delta J_{y_0}(u)]$

and by (a) we have $A_{y_0}(\alpha u + b) = Q_{y_0}(\alpha u + b) = \alpha Q(u) + Q_{y_0}(b)$
$= \alpha A(u) + A(b)$

[we actually get $Q(\alpha u) = \alpha Q(u)$ by Rosenthal's 'Easy Fact' prev sheet]

Step 2 we have y_0 fixed and now fix $\hat{u}$

For these, define $f_{y_0,u} = f: (-r, r) \longrightarrow \mathbb{R}$
$t \longmapsto J(y_0 + t\hat{u})$

we want to show $\boxed{f'(t) = \delta J_{y_0+tu}(u)}$

$$f'(t) = \lim_{s\to 0}\frac{f(t+s) - f(t)}{s} = \lim_{s \to 0}\left[\frac{J(y_0 + (t+s)u) - J(y_0+tu)}{s}\right] = \lim_{s\to 0}\left[\frac{J((y_0+tu)+su) - J(y_0+tu)}{s}\right] = \delta J_{y_0+tu}(u)$$

Step 3 Then we can apply ordinary 1-var MVT to express $J(y_0+tu) - J(y_0) = [\delta J_{y_0+\xi_u u}(u)]\,t$

Because $J(y_0+tu) - J(y_0) = f(t) - f(0) = f'(\xi_u)(t - 0)$ for some $\xi_u \in (0,t)$ — depends on u but that will not matter.

Step 4 Hence

$$\left.\begin{array}{l} \underbrace{J(y_0+tu) - J(y_0)}_{\delta J_{y_0+\xi_u u}(u)\, t} - \underbrace{A_{y_0}(tu)}_{t\,\delta J_{y_0}(u)} \quad \leftarrow \text{def of A} \\ = [\delta J_{y_0+\xi_u u}(u) - \delta J_{y_0}(u)]\,t \end{array}\right\} \Rightarrow \delta J_{y_0+\xi_u u}(u) - \delta J_{y_0}(u) = \frac{J(y_0+tu) - J(y_0) - A(tu)}{t}$$

Step 5 Put it all together:

By hypoth (b), Given $\varepsilon > 0$, $\exists\, \delta > 0$ (indep of $\hat{u}$) $\|(y_0 + t\hat{u}) - y_0\| = \|t\hat{u}\| < \delta$ $\Rightarrow$ For any allowed value of t: $|\delta J_{y_0+tu}(u) - \delta J_{y_0}(u)| < \varepsilon$
$= |t| < \delta$

But $|\xi_u| < |t|$ from step 3, whatever ξ is, it is small enough to be an allowed value

$\Rightarrow |\delta J_{y_0+\xi_u u}(u) - \delta J_{y_0}(u)| < \varepsilon$

$\overset{\text{step 4}}{=} \left|\frac{J(y_0+t\hat{u}) - J(y_0) - A_{y_0}(tu)}{t}\right| < \varepsilon$

we can mult by $\frac{1}{\|\hat{u}\|} = 1$

$$\left|\frac{J(y_0+t\hat{u}) - J(y_0)}{\|tu\|} - \frac{A_{y_0}(tu)}{\|tu\|}\right| < \varepsilon$$

Then since δ indep of $\hat{u}$, let $h := tu$

$\|tu\| < \delta$

is $\|h\| < \delta \Rightarrow \left|\frac{J(y_0+th) - J(y_0) - A_{y_0}(h)}{\|h\|}\right| < \varepsilon$

QED

[We could also have established Rosenthal's [redacted] version using his $\underline{\text{uniform limit}}$ of $\lim_{t\to 0}\frac{1}{t}[J(y+th) - J(y)]$ and not had hypoth (b).]

example 3 p.3

Let $f: \overset{E}{C^1[0,1]} \longrightarrow \overset{F}{C^0[0,1]}$, $u \longmapsto f_u$ where $f_u(t) = u'(t) + t u^2(t)$

Find Df_u

First calculate: $f(u+h) - f(u) = f_{u+h} - f_u = (u+h)' + t(u+h)^2 - [u' + tu^2]$

$= \cancel{u'} + h' + t(\cancel{u^2} + 2uh + h^2) + - \cancel{u'} - \cancel{tu^2}$

$= h' + 2tuh + th^2$

Here we change notation and call 'A' as 'L'.

Take $(L_u h)(t) := h'(t) + 2t\,u(t)\,h(t)$

$L_u h = h' + 2\,\mathrm{id}\,u\,h$ where $\mathrm{id}: \mathbb{R} \to \mathbb{R}$, $t \mapsto t$ is the identity map; we can abuse notation and write this as 't'.

$= [D_t(\cdot) + 2\,\mathrm{id}\,u\,I_{C[0,1]}(\cdot)]h$

We must show $L_u: E \to F$, $h \to L_u h$ is Bdd Linear: $\|L_u h\|_F \le M \|h\|_E$

$\|L_u h\|_{C^0} \le \|h'\|_{C^0} + \underbrace{\|2tuh\|_{C^0}}_{\le 2 \underbrace{\|\mathrm{id}\|_{C^0}}_{1} \|u\|_{C^0} \|h\|_{C^0}} = \underbrace{\|h'\|_{C^0}}_{\le \|h\|_{C^0} + \|h'\|_{C^0}} + \underbrace{2\|u\|_{C^0}\|h\|_{C^0}}_{\le 2\|u\|(\|h\| + \|h'\|)}$

$= \|h\|_{C^1} + 2\|u\|_{C^0}\|h\|_{C^1}$

recall $\|\theta\|_{C^1} = \|\theta\|_{C^0} + \|\theta'\|_{C^0}$

sup norm: $\sup_{x \in [0,1]} \{|\theta(x)|\}$

$\Rightarrow \|L_u h\|_{C^0} \le \underbrace{[1 + 2\|u\|]}_{M} \|h\|_{C^1}$

Then $\dfrac{\|f(u+h) - f(u) - L_u h\|_{C^0}}{\|h\|_{C^1}} = \dfrac{\|\mathrm{id} \cdot h^2\|_{C^0}}{\|h\|_{C^1}} \le \dfrac{1\,\|h^2\|_{C_0}}{\|h\|_{C^0} + \|h'\|_{C^0}} \le \dfrac{\|h\|_{C^0}\cancel{\|h\|_{C^0}}}{\cancel{\|h\|_{C^0}}}$

so $\lim_{\|h\| \to 0} [\;\cdot\;] = \lim_{\|h\|_{C^1} \to 0} \|h\|_{C^0} = 0$ $\boxed{\text{QED}}$

Cheney p.4-2 ex 4

$f: C[0,1] \longrightarrow C[0,1]$, $x \longmapsto \varphi \circ x$ where $\varphi \in C^1(\mathbb{R} \to \mathbb{R})$, $(f(x))(t) = \varphi(x(t))$. Find Df_x

For any $t \in [0,1]$

Real number $\longrightarrow [f(x+h) - f(x)](t) = \varphi(x(t) + h(t)) - \varphi(x(t))$

$= \varphi'(x(t) + \theta_t h(t)) \cdot h(t)$ for some $\theta_t \in (0,1)$ by ordinary MVT in $\mathbb{R}$

So we define a fcn θ ptwise for each t.

Claim $Ah = \varphi'(x(\cdot)) \cdot h(\cdot)$

Want to show: Given $\varepsilon > 0$, $\exists \delta > 0$ $\|h\|_{C^0} = \|h\|_\infty < \delta \Rightarrow \|f(x+h) - f(x) - A_x h\|_\infty < \varepsilon \|h\|_\infty$

Then $\|f(x+h) - f(x) - Ah\| = \|\varphi'(x + \theta h)h - \varphi'(x)h\|_\infty \le \|\varphi'(x+\theta h) - \varphi'(x)\|_\infty \|h\|_\infty < \varepsilon \|h\|_\infty$ because:

φ' is cont so $|\varphi'(y+\lambda) - \varphi(y)| < \varepsilon$ if $|\lambda| < \delta_{\varphi'}$

Thus for $|\theta h| \le 1|h| \le \|h\|_\infty < \delta$ we have the result.

$\square$

Cheney ch 4-1 Prob 1

Let $f: C[0,1] \longrightarrow C[0,1]$
$x \longmapsto f(x)$ where $(f(x))(t) = \int_0^1 g(t, x(s))\,ds$ where $g: \mathbb{R}^2 \to \mathbb{R}$ and 2^{nd} partial g_{yy} is cont.

Find Df_x.

This is another example of using the MVT to do it.

$$f(x+h)(t) - f(x)(t) = \int_0^1 g(t, x(s)+h(s))\,ds - \int_0^1 g(t, x(s))\,ds = \int_0^1 \underbrace{[g(t, x(s)+h(s)) - g(t, x(s))]}_{(D_2 g)(t,\, x(s)+\theta_s h(s))\cdot h(s)} ds$$

For each $s \in [0,1]$ we can write $\longrightarrow$ using ordinary MVT in $\mathbb{R}$, some $\theta_s \in (0,1)$

Claim: $(Ah)(t) := \int_0^1 (D_2 g)_{(t, x(s))} h(s)\,ds$

Then
$$\left| f(x+h)(t) - f(x)(t) - Ah(t) \right| = \left| \int_0^1 \underbrace{(D_2 g)_{(t,\, x(s)+\theta_s h(s))} h(s) - (D_2 g)_{(t, x(s))} h(s)}_{[(D_2^2 g)(t,\, x(s)+\tau_s \theta_s h(s)) h(s)]\, h(s)}\, ds \right|$$

apply MVT again $\rightarrow$ $\tau_s \in (0,1)$

$$\leq \int \left| (D_2^2 g)(t, x(s) + \tau_s \theta_s h(s))\, h^2(s) \right| ds$$

$$\leq \|h\|_\infty^2 \int_0^1 \underbrace{\max_{s \in [0,1]} \left| D_2^2 g_{(t,\, x(s)+\tau_s\theta_s h(s))} \right|}_{\text{Call this number } M} ds$$

$$\Rightarrow \frac{\| f(x+h) - f(x) - Ah \|_\infty}{\|h\|_\infty} \leq \frac{\|h\|_\infty^{\cancel{2}} M \cdot 1}{\cancel{\|h\|_\infty}} \longrightarrow 0_{\mathbb{R}} \text{ as } \|h\|_\infty \to 0 \qquad \square$$

cheney Prob 2 Let $f: \underset{\text{Hilbert}}{X} \longrightarrow \mathbb{R}$ be Frechet dif'b. Prove $\exists\, V_x$ (gradient of f) so $Df_x(h) = \langle V_x, h \rangle$

By def, Df_x is Bdd Linear fcnal, so Riesz Rep thm says $\exists\, V_x$ where $Df_x(h) = \langle V_x, h\rangle\ \forall h \in X$

Compute this for $f: X \longrightarrow \mathbb{R}$
$x \longmapsto \langle x, a \rangle^2$

$$f(x+h) - f(x) = [\langle x+h, a\rangle^2] - \langle x, a\rangle^2$$
$$= [\langle x, a\rangle + \langle h, a\rangle]^2 - \langle x, a\rangle^2 = \cancel{\langle x, a\rangle^2} + 2\langle x, a\rangle\langle h, a\rangle + \langle h, a\rangle^2 - \cancel{\langle x, a\rangle^2}$$

Take $Ah := \langle h, 2\overline{\langle x, a\rangle} a\rangle$

Cauchy-Schwartz in X

Then
$$\frac{\| f(x+h) - f(x) - Ah \|_{\mathbb{R}}}{\|h\|_X} = \frac{\| \langle h, a\rangle^2 \|_{\mathbb{R}}}{\|h\|_X} = \frac{|\langle h, a\rangle|^2}{\|h\|_X} \leq \frac{\|h\|_X^2 \|a\|_X^2}{\|h\|_X} \longrightarrow 0$$

$$Df_x(h) = \langle h, \underbrace{2\overline{\langle x, a\rangle} a}_{\text{This is } grad(f) = V_x} \rangle$$

$\square$

grad depends on inner product, while Df_x does not.

$$f: X^{nls} \longrightarrow Y^{nls}$$

Thm $f = \text{const}\ [f(x) = c\ \forall x] \Rightarrow Df_x = 0_{\mathcal{L}(X \to Y)}$

pf. $\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = \frac{\|c - c - 0h\|}{\|h\|} = \frac{0}{\|h\|} = 0$ so $A = 0$ works and derivs are unique $\square$

Thm (converse) $\left.\begin{array}{l}\mathcal{D} \text{ open Conn subset of } X \\ Df_x = 0\end{array}\right\} \Rightarrow f = \text{const on } \mathcal{D}$ cheney ch4.2 p.6

if $\mathcal{D}$ not Conn, we can have diffrent values for f on diffrent Conn components.

pf. Since f is dif'b in $\mathcal{D}$, it is Cont in $\mathcal{D}$.

we will employ the Classic DisConn argument. Choose $x_0 \in \mathcal{D}$ $f(x_0) = c \in Y$

Let $\mathcal{A} := f^{-1}(c)$. choose any $x \in \mathcal{A}$ (possibly $x = x_0$)

Since $\mathcal{D}$ is open, there is an open ball $B(x,\varepsilon) \subseteq \mathcal{D}$. choose $y \in B$

The line segment $S := \{(tx + (1-t)y \mid t \in [0,1]\}$ is contained in $\mathcal{D}$.

By the "Mean Value Bounds Thm" (coming up shortly) $\|f(x) - f(y)\| \leq \sup_{t\in[0,1]} \|Df_{(tx+(1-t)y)}\| \cdot \|x-y\|$ $= 0$

Thus $f(y) = f(x) = c$ and $\mathcal{A}$ is open in $\mathcal{D}$ (it contains only interior pts).

Suppose $\exists$ any pt $z \in \mathcal{D} \ni f(z) \neq c$

Let $\mathcal{B} = \{\text{all } x \in \mathcal{D} \mid f(x) \neq c\} = f^{-1}(Y - \{c\})$ $Y - \{c\}$ is an Open Set
f cont $\Rightarrow f^{-1}(\text{open}) = \text{open}$

So $\mathcal{A}$ open in $\mathcal{D}$, $\mathcal{B}$ open in $\mathcal{D}$, $\mathcal{A} \cup \mathcal{B} = \mathcal{D}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$

$\Rightarrow \mathcal{D}$ is disconnected $\Rightarrow\Leftarrow$ since $\mathcal{D}$ is Conn by hypoth

$\Rightarrow \mathcal{B} = \emptyset \Rightarrow f(x) = c\ \forall x \in \mathcal{D}$ $\boxed{\text{QED}}$

Thm $\left.\begin{array}{l} f = L \text{ Bdd Linear map} \\ \text{i.e. } f(x+y) = L(x+y) = Lx + Ly \text{ and } L \text{ itself does not depend on } x\end{array}\right\} \Rightarrow Df_x(\cdot) = L(\cdot)\ \forall x$

Remember Bdd Linear = Continuous for linear maps

Pf. $\|L(x+h) - L(x) - Ah\| = \|Lx + Lh - Lx - Ah\| = \|(L-A)h\| = 0$ if $A = L$ and Df_x is unique. $\square$

COR f affine map: $f(x) = Lx + b \Rightarrow Df_x = L$

Thm The operator D is linear: $D(\alpha f + g)_x = \alpha Df_x + Dg_x$

Pf. want: $\lim_{h\to 0} \frac{1}{\|h\|} \|\alpha f(x+h) + g(x+h) - (\alpha f(x) + g(x)) - (\alpha A_x + B_x)\| = 0$

observe $\left\| \frac{\alpha f(x+h) - \alpha f(x) - \alpha A(x)}{|h|} + \frac{g(x+h) - g(x) - B(x)}{|h|} \right\| \leq \| \cdot \| + \| \cdot \| \to 0$

$\square$

Mean Value Thms

Original: $f: \mathbb{R} \to \mathbb{R}$ C^{otd} $\Rightarrow$ $f(b) - f(a) = f'(\xi)(b-a)$ for some $\xi \in (a,b)$

This type of MVT does not generalize to settings where range is not $\mathbb{R}$ (but we do have 'bounds' MVTs)

Counter-example $f: \mathbb{R} \to \mathbb{R}^2$, $t \mapsto \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ Is there some $\xi \in (0, 2\pi)$ $\ni$ $f(2\pi) - f(0) = Df_\xi(2\pi - 0)$?

That would require $\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin \xi \\ \cos \xi \end{bmatrix}(2\pi)$

This is impossible because no ξ makes $\sin \xi = 0$ AND $\cos \xi = 0$.

MVT ($\mathbb{R} \to$ n.l.s.) So it's a curve

$$\left.\begin{array}{l} f: [a,b] \to Y^{nes} \\ Df_t \text{ exists } \forall t \\ \|Df_t\|_{op} \le M \ \forall t \end{array}\right\} \Rightarrow \|f(b) - f(a)\| \le M(b-a)$$

Rudin POMA p.113 gives a simple pf if Y has inner prod.

Cheney's pf: Step 1 It is enough to show this for any α, β satisfying $a < \alpha < \beta < b$ because if $\|f(\beta) - f(\alpha)\| \le M(b-a)$ we can take a seq $(\alpha_i) \searrow a$ and using the continuity of f

$$\lim_{i\to\infty} \|f(\beta) - f(\alpha_i)\| \le \lim_{i\to\infty} M(b-a) \Rightarrow \|f(\beta) - f(\underbrace{\lim \alpha_i}_{a})\| \le M(b-a)$$

and do the same arg for $(\beta_i) \nearrow b$.

And it is enough to show this: for any $\varepsilon > 0$ we have $\|f(\beta) - f(\alpha)\| \le (M+\varepsilon)(b-a)$ Let $\varepsilon \searrow 0$.

Let $\varepsilon > 0$ be given.

Step 2: Define $S := \{ \text{all } t \in [\alpha, \beta] \mid \|f(t) - f(\alpha)\| \le (M+\varepsilon)(t - \alpha) \}$

S is not empty because at least $\alpha \in S$

S is clsd in $[a,b]$ because if $(t_i) \in S$ and $(t_i) \to t$ then $t \in S$:

$$\lim_{i\to\infty} \|f(\underbrace{t_i}_{t}) - f(\alpha)\| \le (M+\varepsilon)(\underbrace{\lim t_i}_{t} - \alpha) \Rightarrow t \in S$$ S contains lim pts.

Step 3: Let $x = \sup(S)$ $x \in S$ since S closed. $\Rightarrow \|f(x) - f(\alpha)\| \le (M+\varepsilon)(x - \alpha)$ (★★)

Claim: $x = \beta$ Suppose $x < \beta$ and get a contradiction

f is difb at x $\Rightarrow$ For our $\varepsilon > 0$, $\exists \delta > 0 \ni |h| < \delta \Rightarrow \|f(x+h) - f(x) - Df_x h\| < \varepsilon |h|$

By shrinking δ if nec, take $\delta < \beta - x > 0$ Take $h = \delta/2$ and define $u := x + h$ ($h = u - x$)

Then we have $\|f(u) - f(x) - Df_x(u-x)\| \le \varepsilon(u - x)$

Reverse Δ ineq: $\|f(u) - f(x)\| - \underbrace{\|Df_x(u-x)\|}_{< M(u-x)} \le \varepsilon(u-x)$ — subtracting something bigger

$$\Rightarrow \|f(u) - f(x)\| \le (M+\varepsilon)(u-x) \quad (★)$$

Step 4: $\|f(u) - f(\alpha)\| = \|f(u) - f(x) + f(x) - f(\alpha)\| \le \underbrace{\|f(u) - f(x)\|}_{(★)} + \underbrace{\|f(x) - f(\alpha)\|}_{(★★)}$

$\le (M+\varepsilon)(u-x) + (M+\varepsilon)(x-\alpha) = (M+\varepsilon)(u-\alpha)$

This shows $u \in S$ but $u > x$ and $x = \sup(S)$ $\Rightarrow\!\Leftarrow$

So we must conclude $x = \beta$

QED

In Koch's Applied Math II class notes, he lists an alternate pf (fiendishly tricky) (4b)
4/8/93

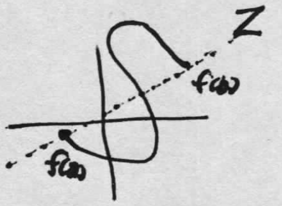
pf (1) For any $u \in Y$, $Z := \text{Span}\{u\}$ is a subsp and we can define a fcnal

$g : Z \longrightarrow \mathbb{R}$ $\quad$ g is Bdd Linear, i.e. Cont

$cu \longmapsto c\|u\|$ when $c \in \mathbb{R}$

By the Hahn-Banach Extension Thm $\exists$ cont linear fcnal $\tilde{g} : Y \longrightarrow \mathbb{R}$ such that

(i) $\tilde{g}(z) = g(z) \ \forall z \in Z$ $\quad$ (ii) $\|\tilde{g}\|_Y = \|g\| = 1$



(2) Now make a clever choice for u: $u = f(b) - f(a)$

Then $\tilde{g}(f(b) - f(a)) = \|f(b) - f(a)\|$

(3) Define $\gamma(t) := \tilde{g}(f(t))$ [this is why we need H-B extension because $f(t)$ not nec in Z]

we are going to use ordinary calculus MVT on γ

⊛ $\gamma(b) - \gamma(a) = \gamma'(\xi)(b-a)$ for some $\xi \in (a,b)$

Now we can turn this expression into what we want:

(4) $\gamma(b) - \gamma(a) = \tilde{g}(f(b)) - \tilde{g}(f(a)) = \tilde{g}(f(b) - f(a))$ since $\tilde{g}$ linear

$= \|f(b) - f(a)\|$ by def of $\tilde{g}$

(5) Show $\gamma'(t) = \tilde{g}(f'(t))$:

$$\gamma'(t) = \lim_{s \to 0} \frac{1}{s}[\gamma(t+s) - \gamma(t)] = \lim_{s \to 0} \frac{1}{s}[\tilde{g}(f(t+s)) - g(f(t))]$$

$$= \lim_{s \to 0} \left[\tilde{g}\left(\frac{f(t+s) - f(t)}{s}\right)\right] \quad \tilde{g} \text{ linear}$$

$$= \tilde{g}\left(\lim_{s \to 0}\left[\frac{f(t+s) - f(t)}{s}\right]\right) \quad \tilde{g} \text{ cont}$$

$$= \tilde{g}(f'(t))$$

(6) So now we have converted ⊛ into

$$\|f(b) - f(a)\| = \tilde{g}(f'(\xi))(b-a)$$

$$\leq \|\tilde{g}(f'(\xi))\|\underset{\text{pos}}{(b-a)} \leq \underbrace{\|\tilde{g}\|_{op}}_{1}\|Df_\xi\|(b-a) \leq \overbrace{\sup_{\xi \in [a,b]}\|Df_\xi\|}^{M}(b-a)$$

□

Koch also has a part (b):

$$\|f(b) - f(a) - Df_t(b-a)\| \leq (b-a)\sup_{s \in [a,b]}|Df_s - Df_t| \quad \forall t \in [a,b]$$

pf. apply MVT we just proved to $\Psi(\underset{s}{t}) := f(\underset{s}{t}) - f(a) - (\underset{s}{t} - a)Df_t$

MVT (Line seg in n.l.s. X $\longrightarrow$ n.l.s. Y)

$$\left.\begin{array}{l} f: \mathcal{U} \subset X \longrightarrow Y \\ \text{Line seg } S = \{\text{all } ta + (1-t)b \mid 0 \le t \le 1\} \subset \mathcal{U} \\ Df_x \text{ exists } \forall x \in S \end{array}\right\} \Rightarrow \|f(b) - f(a)\| \le \overbrace{\sup_{x \in S} \|Df_x\|}^{M} \, |b-a|$$

Pf apply the MVT ($\mathbb{R} \to$ nls)

Define $g: [0,1] \longrightarrow Y$

$$t \longmapsto f(\underbrace{ta + (1-t)b}_{\ell(t)}) = (f \circ \ell)(t)$$

Chain rule $Dg_t = Df_{\ell(t)} D\ell_t$ ($\ell'(t)$)

$D\ell_t: \mathbb{R} \longrightarrow X$, $h \longmapsto (a-b)h$

Then by prev Thm $\|g(1) - g(0)\| \le \|Dg_t\|_{op}(1-0)$

$$\|f(b) - f(a)\| \le \|Df_{\ell(t)}\|_{op} \|D\ell_t\| \, 1 \le M \|b-a\| \qquad \square$$

Special Case (Line seg in $X \longrightarrow \mathbb{R}$)

$$\left.\begin{array}{l} f: X \longrightarrow \mathbb{R} \\ S = \{\text{all } (ta + (1-t)b) \mid t \in [0,1]\} \end{array}\right\} \Rightarrow \exists \text{ pt } z \in S \text{ such that } f(b) - f(a) = Df_z(\vec{a} - \vec{b})$$

should be $tb + (1-t)a$

pf. again $g: [0,1] \longrightarrow \mathbb{R}$, $t \longmapsto f(\ell(t))$

But here we can apply the ordinary MVT (1-var) to g: $\exists\, \xi \in (0,1) \ni g(1) - g(0) = g'(\xi)(1-0)$

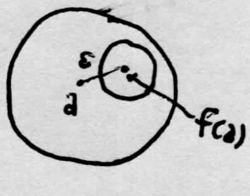
$$\Rightarrow \underset{a}{f(\ell(1))} - \underset{b}{f(\ell(0))} = Df_{\ell(\xi)}(\ell'(\xi)) = Df_z(a-b) \qquad \square$$

Koch's favorite <u>Contraction Mapping Thm using MVT</u> from notes 4/15/93. He claims this works when others don't.
See also discussion in L&S p.229 but no MVT there.

Thm

$$\left.\begin{array}{l} f: X \longrightarrow X \text{ difb } (X \text{ Banach}) \\ \|F(a) - a\| < \varepsilon \ (a \text{ is close to being a fixed pt}) \\ \|Df_x\| \le \lambda < 1 \ \forall x \in \bar{B}(a,r) \ [Df_x \text{ small near } a] \\ \text{Relation among params: } \frac{\varepsilon}{1-\lambda} \overset{!}{<} r \end{array}\right\} \Rightarrow \left.\begin{array}{l} 1.\ f(\bar{B}) \subseteq \bar{B} \text{ (no translations allowed)} \\ 2.\ f \text{ satisfies contraction mapping cond:} \\ \quad \|f(x) - f(y)\| \le \theta \|x-y\| \\ \quad 0 < \theta < 1 \end{array}\right\} \Rightarrow \text{By Contract Map Thm, } \exists!\ \text{FP } x^* \in \bar{B}$$

pf. $f(\bar{B}) \subseteq \bar{B}$. Choose any $x \in \bar{B}(a,r)$



$$\|f(x) - a\| = \|f(x) - f(a) + f(a) - a\| \overset{\text{Triang}}{\le} \overbrace{\|f(x) - f(a)\|}^{g(x)} + \underbrace{\|f(a) - a\|}_{< \varepsilon \text{ by hypoth}}$$

$$\overset{\text{MVT}}{\le} \sup \|Df_\xi\| \, \|x - a\| + \varepsilon$$

For ξ on line segment from x to a

$$\le \sup_{\text{any } z \in \bar{B}} \|Df_z\| \, r$$

$$\le \lambda r + \varepsilon \Rightarrow \text{is } \lambda r + \varepsilon < r? \text{ yes if } \frac{\varepsilon}{1-\lambda} < r \qquad \square$$

For contraction cond: $x, y \in \bar{B}$

$$\|f(x) - f(y)\| \le \sup_{\xi \in S} \|Df_\xi\| \, \|x-y\|$$

$$\le \lambda \|x-y\|$$

$\boxed{\text{QED}}$