$C^1[a,b]$ is complete with the norm $\|f\|_{C^1} = \|f\|_{sup} + \|f'\|_{sup}$.
[so Phil Schafer was wrong! And I was wrong to believe him!]

pf Let (f_n) be a cauchy seq in $C^1[a,b]$
i.e. Given $\varepsilon>0$, $\exists N \ni n>m>N \Rightarrow \|f_n - f_m\|_{C^1} < \varepsilon$
ie $\|f_n - f_m\|_\infty + \|f_n' - f_m'\|_\infty < \varepsilon$

Then since $C[a,b]$ is complete,
$\exists f \in C[a,b] \ni f_n \to f$ pt wise
$\exists g \in C[a,b] \ni f_n' \to g$ pt wise

We need to show $g = f'$ and then we are done; $f \in C^1$
Play the "$\varepsilon/3$ game" (well, I'll be sloppy)

For any $x \in [a,b]$
$$\left| f(x) - f(a) - \int_a^x g(t)\,dt \right| = \left| f(x) - f_n(x) + f_n(a) - f(a) + \int_a^x (f_n' - g)\,dt \right|$$

cont fcn cpt set $\Rightarrow$ unif conv over $[a,b]$ $\longrightarrow$
$$\leq \underbrace{|f(x) - f_n(x)|}_{<\varepsilon} + \underbrace{|f_n(a) - f(a)|}_{<\varepsilon} + \underbrace{\int_a^x |f_n' - g|\,dt}_{<\varepsilon(b-a)}$$
if $n > N_1$ if $n > N_2$

Thus $\left| f(x) - f(a) - \int_a^x g \right| < (2+(b-a))\varepsilon$

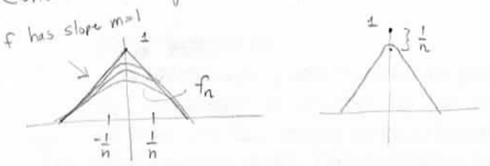
Since ε can be arb small $\underbrace{f(x) = f(a) + \int_a^x g(t)\,dt}$
The LHS is differentiable and has deriv $g'(x)$ by FTOC

$\Rightarrow f(x) = g'(x)$ for arb x.

$\square$

CLEVER COUNTER-EXAMPLE THAT IS WRONG

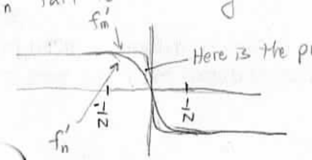
Consider a seq of bump fcns, sharpening to a peak (and symmetric about y axis)



f_n coincides with f except in the region $(-\frac{1}{n}, \frac{1}{n})$ where it smoothly rounds off, lying at most $\frac{1}{n}$ below the peak.

(f_n) looks like a Cauchy seq with $\|\ \|_{C^1}$, and while $\|f_n - f_m\|_\infty < \frac{1}{N}$

The derivs f_n' fail to be Cauchy



Here is the problem: arb close to 0 $\|f_n' - f_m'\|_\infty \approx \frac{1}{2}$ slope of f.

$\square$

Remember $R(a,b) = P_a^b A$, flow operator for $t=a$ to $t=b$.

domain of def of ODE $\downarrow \mathbb{R}$

(*) $\dot{x}(t) = A_t\, x(t)$ $\quad A(t+T) = A(t)$.

Thm F.1 want to show $R(a+T, t+T) = R(a,t)$ $\forall t$.

Pf. From Thm 6.1 p.68 $(D_2 R)(a,t) = A_t R(a,t)$ $\forall$ $a,t \in I \times I$. — see p.69 Can we take $I = \mathbb{R}$?

Then if we plug in "a" $= a+T \in I$, "t" $= t+T \in I$ we see

$\tilde{g}(t) = R(a+T, t+T)$ also satisfies (*)

Now to impose uniqueness, we must append an arb IC to (*): $x(a) = x_0$.

$f(t) = R(a,t)\, x_0$ is the unique solution by Thm 6.2 p. 69, since
$f(a) = R(a,a)\, x_0 = x_0$.

But $g(t) := R(a+T, t+T)\, x_0$ also satisfies (*) and the IC:
$g(a) = R(a+T, a+T)\, x_0 = x_0$

uniqueness $\Rightarrow$ $f(t) = g(t)$ $\forall t$ $\Rightarrow$ $R(a,t)\, x_0 = R(a+T, t+T)\, x_0$
$\Rightarrow$ $R(a,t) = R(a+T, t+T)$ since x_0 was arb.

$\square$

From p.69 Thm 6.2 the unique solution to $\begin{cases} \dot{x} = A_t x \\ x(a) = x_0 \end{cases}$ is $f(t) = P_a^t(A)\, x_0$

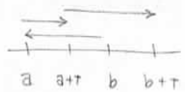
Here we write that as $f(t) = R(a,t)\, f(a)$.

In particular, $f(a+T) = \underbrace{R(a, a+T)}_{C_a} f(a)$. If we want f to be periodic $C_a \overset{!}{=} Id_E$.

Thm F.2 all monodromy operators are conjugate.

$$C_b \overset{def}{=} \underset{P_b^{b+T}}{R(b, b+T)} \overset{\text{chasels'}}{=} \underset{P_{a+T}^{b+T}}{R(a+T, b+T)}\, \underset{P_a^{a+T}}{R(a, a+T)}\, \underset{P_b^a}{R(b,a)} \overset{\text{Thm F.1}}{=} R(a,b)\cdot C_a \cdot R(a,b)^{-1}$$

flow operator $\nearrow P_b^{b+T}$



$\square$

Thm F.3 all solutions of $\dot{x} = A_t x$ have period T $\iff$ $\exists$ at least one $a \in I \ni C_a = Id_E$

($\Rightarrow$) on p.168 we saw, for any solution f, $f(t) = R(a,t) f(a)$ $\forall t$.
If we know $f(t+T) = f(t)$ $\forall t$, then for $t = a$,

$$f(a+T) = f(a)$$
$$\| \quad R(a, a+T) f(a)$$
$$\| \quad C_a f(a) \qquad \Rightarrow \quad C_a = Id_E.$$

($\Leftarrow$) if $C_a = Id_E$ for some $a \in \mathbb{R}$, since $C_s = R(a,s)\, C_a\, R(a,s)^{-1}$
$= R \cdot Id \cdot R^{-1}$
$= Id$

$\Rightarrow C_s = Id$ $\forall s$

Then, since for any t, $f(t+T) = C_t f(t)$ $\left[\begin{array}{l} \text{p.168} \\ f(a+T) = R(a, a+T) f(a) \\ \text{rename "}a\text{" as "}t\text{"} \end{array}\right]$.
$= Id\, f(t)$
$= f(t)$

$\square$

Thm F.4 (1) K is EW of C_a $\Rightarrow$ $\exists$ soln f of $\dot{x} = A_t x \ni f(t+T) = K f(t)$ $\forall t$

(2) $\exists$ nonzero soln of period T $\iff$ $K = 1_{\mathbb{R}}$ is an EW of C_a.

Pf. $C_a : E \longrightarrow E$ Bdd Linear
Let $C_a u = K u$ $\quad u \neq 0_E$.
Let f be soln to $\begin{cases} \dot{x} = A_t x \\ x(a) = x_0 \end{cases}$ want to show: $f(t+T) = K f(t)$ $\forall t$.

$f(a+T) = C_a f(a) = C_a u = K u = K f(a)$

Define $g(t) := f(t+T) - K f(t)$ $\quad$ want: $g(t) \equiv 0$.
$= (f \circ \tau)(t) - K f(t)$

Then g satisfies $\begin{cases} \dot{x} = A_t x \\ x(a) = 0 \end{cases}$ because $g'(t) = f'(\tau(t))\tau'(t) - K f'(t)$
$= f'(t+T) \cdot 1 - K f'(t)$
$= A_{t+T} f(t+T) - K A_t f(t)$
$= A_t [f(t+T) - K f(t)] = A_t g(t)$

But the 0 fcn also satisfies, so by uniqueness p.69 $g = 0$.

END (1).

(2)
($\Leftarrow$) if $K = 1_{\mathbb{R}}$, the above soln f is of period T.
($\Rightarrow$) if f is nonzero soln of period T,
$f(a) = f(a+T) = C_a f(a)$
$f(a)$ is EV of C_a w/ $K = 1$.

QED.

Thm F.5 Given $\dot{x} = A_t x$ with period T, every new eq $\dot{y} = \tilde{A}_t y$ that we can make by a linear COV $y = S_t x$ (of period T) must satisfy $\tilde{C}_a = S_a C_a S_a^{-1}$.

Pf. $y = S_t x \Rightarrow x = S_t^{-1} y$ (another hypothesis was $S_t \in GL(E,E)$).

$x(t) = S_t^{-1} y(t) = \mathrm{Comp}((J \circ S)(t), y(t))$ see p.69 Bopp for details Leibniz rule p.6-7

$\Rightarrow x'(t) = -S_t^{-1} \circ \dot{S}_t \circ S_t^{-1} y + S_t^{-1} \dot{y}$

Plug into $\dot{x} = A_t x$: $S_t^{-1}\dot{y} = S_t^{-1}\dot{S}_t S^{-1} y + A_t S_t^{-1} y$

$\Rightarrow \dot{y} = \underbrace{[\dot{S}_t + SA]S^{-1}}_{\tilde{A}_t} y$

we know S_t and A_t have period T. $\dot{S}_t$ also has period T: $\dot{S}_{(t+T)} = \lim_{h\to 0} \frac{1}{h}(\overset{S_{(t+h)}}{\overset{\shortparallel}{S_{(t+T+h)}}} - \overset{S_t}{\overset{\shortparallel}{S_{(t+T)}}}) = \dot{S}_t$

$\Rightarrow \tilde{A}_t$ also has period T

$\Rightarrow \dot{y} = \tilde{A}_t y$ has monodromy map $\tilde{C}_a = \tilde{R}(a, a+T)$

gen soln: $y(t) = \tilde{R}(a,t)\underbrace{y(a)}_{S(a)x(a)}$

we also know $x(t) = R(a,t)x(a)$ so we get $y(t) = S_t x(t) = S_t R(a,t) x(a)$

$\Rightarrow \tilde{R}(a,t) S_a x(a) = y(t) = S_t R(a,t) x(a)$ $x(a)$ arb

$\Rightarrow \tilde{R} = S_t R S_a^{-1}$

take $t = a+T$ $\tilde{R}(a, a+T) = S_{a+T} \circ R(a, a+T) \circ S_a^{-1}$

$\tilde{C}_a = S_a \cdot C_a \cdot S_a^{-1}$

QED

Given $\dot{x} = A_t x$ w/ period T, $\dot{y} = B_t y$ w/ period T, can we alway find S_t w/ period $T \ni y = S_t x$? $S_t \in GL(E,E)$.

NO; by above thm, every allowable S_t gives $\tilde{C}_a = M C_a M^{-1}$

Presumably for some B_t's the assoc monodromy map is in a different similarity class than C_a.

Thm F.6

E fin dim v.s. over $\mathbb{C}$

$A: \mathbb{R} \longrightarrow \mathcal{L}(E,E)$, $t \longmapsto A_t$ $\left\{\begin{array}{l}\text{continuous}\\ T\text{-periodic}\end{array}\right\} \Rightarrow \exists$ at least one C^1, T-periodic fcn $S: \mathbb{R} \longrightarrow GL(E,E)$, $t \longmapsto S_t$ such that the COV $y = S_t x$ reduces the eq $\dot{x} = A_t x$ to $\dot{y} = By$ where B does not depend on t.

$\boxed{\text{we absorb the time dependence into the co-ord system}}$

Pf.

Let $C_a = R(a, a+T)$ be the monodromy operator assoc w/ $\dot{x} = A_t x$.

we know C_a^{-1} exists because $C_a \overset{p.70}{=} P_a^{a+T}(A)$ the flow map is invertible.

APPENDIX E p.167 states "For any invertible $L \in \mathcal{L}(E_{\mathbb{C}}, E_{\mathbb{C}})$, $\exists$ at least one $B \in \mathcal{L}(E_{\mathbb{C}}, E_{\mathbb{C}}) \ni L = e^B$"

Here $L = C_a$ and we obtain $\hat{B} \ni C_a = e^{\hat{B}}$. Then define $B := \frac{1}{T}\hat{B}$. and $C_a = e^{TB}$. $T = T + 0i \in \mathbb{C}$.

$\triangleright$ Now we simply verify that $S_t := e^{tB} R^{-1}(a,t)$ works:

S_t is C^1 in t since compo of C^1 fcns.

we know from the previous thm that when we make a COV $y = S_t x$ we get $\dot{y} = [\underbrace{(\dot{S}_t + S_t A_t) S_t^{-1}}_{\text{claim: } = B}] y$ and so time independent.

we have to compute $\dot{S}_t$ explicitly. $S_t = \mathrm{Comp}(e^{tB}, \underset{(J \circ R)(a,t)}{\overset{\shortparallel}{R^{-1}(a,t)}})$

using Leibnitz p.6-7 (cf p.69 Bopp for details)

let $F(t) := R(a,t)$

actually it's just the old fashioned prod & chain rules.

$\dot{S}_t \overset{p.53}{=} Be^{tB}R^{-1}(a,t) + e^{tB}(DJ_{F(t)} \cdot DF_t(\cdot))$

$- F_{(t)}^{-1} \cdot \underbrace{DF_t(\cdot)}_{A_t F(t)} \cdot F^{-1}(t)$ p.39

$= Be^{tB}R^{-1}(a,t) - e^{tB}R^{-1}A_t$

Then $(\dot{S}_t + S_t A_t) S_t^{-1} = [(Be^{tB}R^{-1} - \cancel{e^{tB}R^{-1}A_t}) + \cancel{e^{tB}R^{-1}A_t}] Re^{-tB} = B$.

Now we must show S is of period T: want $S_{t+T} = S_t$

$S_{(t+T)} \overset{\text{def}}{=} e^{(t+T)B} R^{-1}(a, t+T) \overset{\text{circa p.70}}{=} e^{(t+T)B} R(t+T, a) = e^{tB} \underbrace{e^{TB}}_{= C_a = R(a, a+T)} R(t+T, a)$

$= e^{tB} P_a^{a+T} P_{t+T}^a$ a a+T t+T

$= e^{tB} R(t+T, a+T) \underset{\uparrow \text{ Thm F.1}}{=} e^{tB} R(t,a) = e^{tB} R^{-1}(a,t) = S_t$

QED.