ex (b)

$$f : \overset{E}{C^1[0,1]} \longrightarrow \overset{F}{C^0[0,1]}$$
$$u \longmapsto f_u \quad \text{where} \quad f_u(t) = u'(t) + t \cdot u^2(t)$$

Then $f(u+h) - f(u) = f_{u+h} - f_u$

where $f_{u+h}(t) - f_u(t) =$

$\cancel{u'(t)} + h'(t) + t\left(\cancel{u^2(t)} + 2u(t)h(t) + h^2(t)\right) - \cancel{u'(t)} - \cancel{tu^2(t)}$

$\mathrm{id} : [0,1] \longrightarrow \mathbb{R}$, $t \longmapsto t$

Take $(L(h))(t) := h'(t) + 2 \cdot t \cdot u(t) \cdot h(t)$

$\Rightarrow L(h) = h' + 2 \cdot \mathrm{id} \cdot u \cdot h$

L is obviously linear, L is cont because "Bdd linear"

$$\|Lh\|_{C^0} \overset{\text{Triang}}{\leq} \|h'\|_{C^0} + 2 \cdot \underbrace{\|\mathrm{id}\|_{C^0}}_{=1} \cdot \|u\|_{C^0} \cdot \|h\|_{C^0}$$

$$\leq \left[1 + 2\|u\|_{C^0}\right] \underbrace{\|h\|_{C^1}}_{\|h\|_{C^0} + \|h'\|_{C^0}}$$

Then
$$\frac{\|f(u+h) - f(u) - L(h)\|_{C^0}}{\|h\|_{C^1}} = \frac{\|\mathrm{id} \cdot h^2\|_{C^0}}{\|h\|_{C^1}} = \frac{\|h^2\|_{C^0}}{\|h\|_{C^0} + \|h'\|_{C^0}}$$

$$\leq \frac{\|h\|_{C^0}^{\cancel{2}}}{\cancel{\|h\|_{C^0}}} \longrightarrow 0 \text{ as } \|h\|_{C^1} \to 0.$$

$\square$

$$b : E_1 \times E_2 \longrightarrow F$$

we compute $b(a+h) - b(a)$

$$= b[(a_1,a_2)+(h_1,h_2)] - b[(a_1,a_2)]$$

$$= b[(a_1+h_1,\ a_2+h_2)] - b[(a_1,a_2)]$$

bilinearity

$$= \cancel{b[(a_1,a_2)]} + \underbrace{b[(a_1,h_2)] + b[(h_1,a_2)]}_{\substack{\text{we take this as} \\ Db_a(h)}} + b[(h_1,h_2)] - \cancel{b[(a_1,a_2)]}$$

Must show $\dfrac{\| b[(h_1,h_2)] \|_F}{\| h \|_{E_1 \times E_2}} \longrightarrow 0_{\mathbb{R}}$ as $\|h\| \to 0$.

Drop the stupid [ ]

By p.150 $\| b(h_1,h_2) \| \le \|b\|_{op} \cdot \|h_1\|_{E_1} \cdot \|h_2\|_{E_2} \xrightarrow{\text{Trivially}} \le \|b\| \left[\|h_1\|\|h_2\| + \|h_1\|^2 + \|h_1\|\|h_2\| + \|h_2\|^2 \right]$

$p>0,\ q>0$

$pq \le (p+q)^2 = p^2 + 2pq + q^2$

$$= \|b\|_{op} \left[\|h_1\|_{E_1} + \|h_2\|_{E_2} \right]^2$$

$$= \|b\| \cdot \|h\|^2_{E_1 \times E_2}$$

$\square$

$p = b \circ \lambda$ $\quad$ $p : \overset{E}{x} \xmapsto{\lambda} \overset{F_1 \times F_2}{(f(x), g(x))} \xmapsto{b} \overset{G}{b(f(x), g(x))}$

chain rule

$Dp_a = Db_{\lambda(a)} \cdot D\lambda_a$

To compute $D\lambda_a$:

$$\lambda : E \longrightarrow F_1 \times F_2$$
$$x \longmapsto (f(x), g(x))$$

$$\lambda(x+h) - \lambda(x) = (f(x+h), g(x+h)) - (f(x), g(x))$$
$$= (f(x+h) - f(x),\ g(x+h) - g(x))$$

Hence I'm going to guess

$D\lambda_a(h)$ $\quad$ "Ah" $= (Df_x(h), Dg_x(h))$.

$$= Db_{(f(a), g(a))} \circ (Df_a(h), Dg_a(h))$$

$$\overset{p.4}{=} b(Df_a(h), g(a)) + b(f(a), Dg_a(h))$$

$\square$

See p.36 for continuation

$$f: E_1 \oplus \cdots \oplus E_n \longrightarrow F$$
$$(x_1, \ldots, x_n) \longmapsto$$

Show: $Df_a(h) = \sum_{r=1}^{n} D_r f_a(h_r)$

work on RHS

$$\downarrow \text{def}$$

$$= \sum D(f \circ I_r^a)_{a_r}(h_r)$$

$$= \sum Df_a \circ \dot{I}_r(h_r)$$

if e.g. $n = 3$

$$= \left[Df_a((h_1,0,0)) + Df_a((0,h_2,0)) + Df_a((0,0,h_3)) \right]$$

$$= Df_a(h) \quad \text{linearity}$$

$$h = (h_1\ h_2\ h_3)$$

$$D_r f_a(\cdot) \overset{\text{def}}{=} D(f \circ I_r^a)_{a_r}(\cdot)$$

$$\overset{\text{chain}}{=} Df_{\underbrace{I_r^a(a_r)}_{=a}} \cdot D(I_r^a)_{a_r}(\cdot)$$

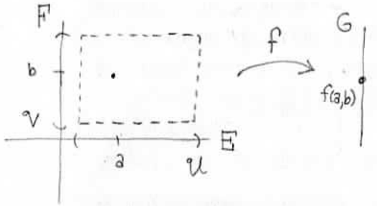
$$= Df_a \cdot \dot{I}_r$$

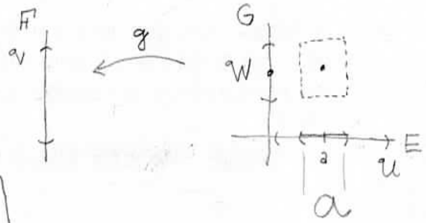
$$[D_1 f \quad D_2 f \quad D_3 f] \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

IMP FCN THM (SEE ALSO COR ↘)

$$\left.\begin{array}{l} E, F, G \quad \text{Banach sp} \\ {}_{a\in}\mathcal{U} \text{ open in } E,\ {}_{b\in}\mathcal{V} \text{ open in } F \\ f: \mathcal{U}\times\mathcal{V} \longrightarrow G \quad \text{class } C^1 \\ D_2 f_{(a,b)} : F \longrightarrow G \quad \text{iso} \end{array}\right\} \Longrightarrow \begin{array}{l} \exists \text{ open sets } \mathcal{A} \text{ containing } a \\ \qquad\qquad\quad \mathcal{W} \quad \text{''} \quad f(a,b) \\ \exists!\ g: \mathcal{A}\times\mathcal{W} \longrightarrow \mathcal{V} \ni \\ f(x, g(x,w)) \equiv w \quad \forall\, (x,w)\in \mathcal{A}\times\mathcal{W} \end{array}$$

The same number deriv as the arg we want to replace.





See also picture for φ ↗

Pf. The whole idea is to construct a fcn $\varphi: \mathcal{U}\times\mathcal{V} \longrightarrow E\times G$ and apply the Inv Fcn Thm to it.
$\langle x, y\rangle \longmapsto \langle x, f(x,y)\rangle$

ie $\varphi := \pi_1 \times f$

Step 1 $\varphi: \pi_1 \times f$.

By direct computation, we see $D\varphi_{(a,b)}(h,k) = \langle h,\ D_1 f_{ab}(h) + D_2 f_{ab}(k)\rangle$:

we need $\dfrac{\| \varphi(a+h, b+k) - \varphi(a,b) - L(h,k)\|}{\|\langle h,k\rangle\|} \longrightarrow 0_{E\times G}$ as $\|\langle h,k\rangle\| \to 0$

$$\varphi(a+h, b+k) - \varphi(a,b) = \langle a+h,\ f(a+h, b+k)\rangle - \langle a, f(a,b)\rangle$$
$$\doteq \langle h,\ f(a+h, b+k) - f(a,b)\rangle$$

Hence we take $L(h,k) := \langle h,\ Df_{(a,b)}(h,k)\rangle$

Then $\dfrac{\| \varphi(a+h,b+k) - \varphi(a,b) - L(h,k)\|}{\|\langle h,k\rangle\|} = \underbrace{\left\| \left\langle 0,\ \dfrac{f(a+h,b+k) - f(a,b) - Df_{ab}(h,k)}{\|(h,k)\|}\right\rangle \right\|}_{\text{since } f \text{ dif'b, this } \longrightarrow 0}$

Step 2 We see that $D\varphi_{(a,b)}$ is invertible by explicitly computing the inverse:

we need a linear map $Q \ni \begin{cases} Q \circ D\varphi_{(a,b)}(h,k) = \langle h,k\rangle \\ D\varphi_{(a,b)} \circ Q(\tilde{h},\tilde{k}) = \langle \tilde{h},\tilde{k}\rangle \end{cases}$

work with first eq: $Q(h,\ D_1 f_{ab}(h) + D_2 f_{ab}(k)) \overset{!}{=} (h,k)$.

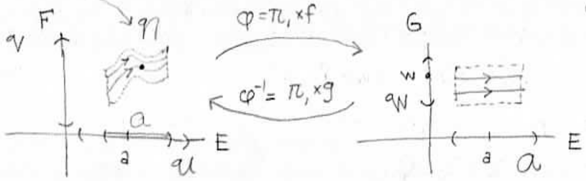
Thus $Q^{①} \overset{!}{=} \pi_1$; For $Q^{②}$ note that if $D_1 f(h) + D_2 f(k) = k'$
Then $k = D_2 f^{-1}(k' - D_1 f(h))$

Thus $Q^{②}(h', k') = D_2 f_{ab}^{-1}(k' - D_1 f_{ab}(h'))$

Explicit formula for $D\varphi_{ab}^{-1}$:

$Q: (h', k') \longmapsto \langle h',\ D_2 f_{ab}^{-1}(k' - D_1 f_{ab}(h'))\rangle$

Step 3 Since $D\varphi_{(a,b)}$ is invertible, Inv Fcn Thm gives us
φ : Nbhd of $(a,b) \xrightarrow{\text{diffeo}}$ nbhd of $\varphi(a,b)$ [By cutting down the size, we can take this to be $\mathcal{A}\times\mathcal{W}$]



Specifically, what we care about is: For every $\langle x, w\rangle \in \mathcal{A}\times\mathcal{W}$
$\exists$ a pt $\langle x, y\rangle \in \mathcal{N} \ni \varphi^{-1}(x,w) = \langle x,y\rangle$

p.6

ie $(\varphi^{-1})^{①}(x,w) = x \longrightarrow (\varphi^{-1})^{①} = \pi_1$
$(\varphi^{-1})^{②}(x,w) = y$

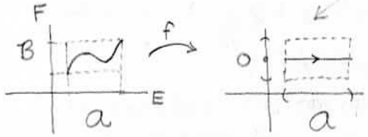
Define $g := (\varphi^{-1})^{②}$ and clearly g is smooth.

COR 3.2

$$\left.\begin{array}{l} \text{Same hypoths as Thm} \\ \text{Fix a particular } w\in\mathcal{W}, \text{ say } w=0 \end{array}\right\} \Rightarrow$$

① $\exists$ nbhd $\mathcal{A}$ of a
$\mathcal{B}$ of b
$\exists!\ C^1$ map $g_0 : \mathcal{A} \longrightarrow \mathcal{B} \ni f(x, g_0(x)) \equiv 0$
$\forall x\in\mathcal{A}$.

② $D(g_0)_x = -D_2 f^{-1}_{(x,g(x))} \circ D_1 f_{(x,g(x))}$



By def of $E_2 = \{{}^{\text{all}}\, h\in E \mid Df_a(h) \neq 0\}$

we show $D_2 f_a$ is an iso: Recall from p.8 $D_2 f_a = Df_a \circ \mathcal{I}_2$

Since $E = E_1 \oplus E_2$, if $h\in E$, $h = h_1 + h_2 = \langle h^{①}, 0\rangle + \langle 0, h^{②}\rangle$

▷ clearly $D_2 f_a$ is linear

▷ $D_2 f_a$ is Onto F: Since Df_a is onto, Given $v\in F$ $\exists$ $h\in E \ni$

$$Df_a(h) = v$$
$$\overset{\shortparallel}{Df_a}[\langle h^{①}, 0\rangle + \langle 0, h^{②}\rangle]$$
$$= 0 + Df_a(0, h^{②})$$
$$\Rightarrow D_2 f_a(h^{②}) = v$$

▷ $D_2 f_a$ is One-to-One since $h\in E_2 \Rightarrow Df_a(h) \neq 0$
$D_2 f_a(h) \neq 0$
$\ker(D_2 f_a) = \{0\}$

□

what we are eventually showing is that the nonlinear ODE defined on $[0,1]$ & by $u'(t) + t \cdot (u(t))^2 = g(t)$ has a soln $u \in C^1[0,1]$ if $|g(t)| < \varepsilon \ \ \forall t$.

Define

$$f : \overset{E}{C^1[0,1]} \longrightarrow \overset{F}{C^0[0,1]}$$
$$u \longmapsto f_u \quad \text{where } f_u(t) = u'(t) + t\,u^2(t).$$

From p. 3 we know $df_u(h) = h' + 2tuh$

$0 \in C^1[0,1] \longrightarrow \Rightarrow df_0(h) = h' = \frac{d}{dt}(h)$.

df_0 is onto $C^0[0,1]$: choose any $g \in C^0[0,1]$
"$\frac{d}{dt}$"
$\exists\, G \ni G' = g$, namely $G(t) = \int_0^t g(s)\,dt$.

Let $E_1 := \ker(df_0) = \{\text{all } h \in C^1 \mid df_0(h) = 0_{C^0}\} = \{\text{all const fcns?}\}$

$E_2 := \{\text{all } h \in C^1 \mid \int_0^1 h\,dt = 0\}$

CLAIM: $E = E_1 \oplus E_2$

pf: For any h, we can write $h = \underbrace{h - c}_{h_2} + \underbrace{c}_{h_1}$ where $c = \int_0^1 h\,dt$ □

E_2 is topologically clsd subsp because if $h_k \to h \in C^1$, want $\int_0^1 h = 0$.

$\Rightarrow \sup_{[0,1]} |h_k(t) - h(t)| + \sup |h_k'(t) - h'(t)| < \bar{\varepsilon}$ if $k > K$.

$\Rightarrow |h_k(t) - h(t)| < \bar{\varepsilon} \ \ \forall t$

$\Rightarrow \left|\int_0^1 h_k - h\right| \le \int_0^1 |h_k - h| < \bar{\varepsilon}(1-0) = \bar{\varepsilon}$

$\left|\int_0^1 \overset{0}{h_k} - \int_0^1 h\right| \Rightarrow \int_0^1 h\,dt < \bar{\varepsilon} \ \ \forall \bar{\varepsilon}$

GOTO TOP ↗

Now we can apply Thm 3.3:

Thm 3.3: $f : \overset{E}{\mathcal{U}} \to F \ \ C^1$
df_a onto F for some $a \in \mathcal{U}$
$\exists$ top clsd subsp $E_2 \ni E = \ker(df_a) \oplus E_2$
$\Rightarrow$ $f(\mathcal{U})$ contains an open ball around $f(a)$.

Here: "a" $= 0_{C^1[0,1]}$

"$f(a)$" $= f(0_{C^1}) = f_0$ where $f_0(t) = 0'(t) + t + 0^2(t) = 0_{\mathbb{R}}$
$= 0_{C^0}$

So by Thm 3.3 $\exists$ a ball $B(\overset{f_0}{0_{C^0}}, \varepsilon)$ in $C^0[0,1] = F \ni$
if $g \in B(0_{C^0}, \varepsilon)$ then $g \in f(\mathcal{U})$ [take $\mathcal{U} = E = C^1[0,1]$]

$B(0_{C^0},\varepsilon) = \{\text{all } g \mid \sup_{t\in[0,1]} |g(t) - 0| < \varepsilon\}$

But if $g \in f(C^1[0,1])$, then $\exists\, u \in C^1[0,1] \ni$
$f(u) = g$
ie $f_u = g$
ie $u'(t) + t\,u^2(t) = g(t) \ \ \forall t \in [0,1]$.

This is saying the ODE has a soln. □