$\dot{x} = f(x) \qquad f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\varphi : \mathbb{R} \to \mathbb{R}^n$ is a soln if $\dot{\varphi}(t) = f(\varphi(t)) \ \forall t \in \mathbb{R}$.

so follow L&S ch 9 we have $g : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $(t, x) \longmapsto g_t(x) = g(t,x) = g_x(t)$

" g_x " is a " φ " that passes thru pt x at $t = 0$; $g_x(0) = x$.

we can expand g_x in a Taylor poly. about $t_0 = 0$:

$$\underset{\underset{g_t(x)}{\parallel}}{g_x(t)} = \underset{\underset{x}{\parallel}}{g_x(0)} + \underset{\underset{f(g_x(0))}{\parallel \text{ ode}}}{\dot{g}_x(0)}(t-0) + \frac{\ddot{g}(\xi)}{2}(t-0)^2$$

$$\Rightarrow g_t(x) = x + f(x)t + O(t^2) \qquad (\star)$$

<u>Thm 2</u> $\underset{\vec{\nabla}\cdot\vec{f}}{\operatorname{div}(f)} = 0 \Rightarrow g_t$ preserves Leb ⓜ: $V(t) = V(0)$.

Let D_0 be a ⓜable set in $\mathbb{R}^n$.

pf.

<u>Lemma 1</u>: $\dot{V}(0) = \int_{D_0} \operatorname{div}(f)\, dx$

pf. By def, $V(t) := \int_{g_t(D_0)} 1\, dy \overset{\substack{\text{Apostol MA p. 421} \\ \text{cov formula}}}{=} \int_{D_0} |\det(d(g_t)_x)|\, dx$

Q. But this requires that g_t is a diffeo

A. we have this because we assumed g_t is defed $\forall t$; its in a 1 param group of diffeos!

Q Are we assuming then that the streamlines never cross?

Now from $(\star)$, $d(g_t)_x = I + t \cdot df_x + O(t^2)$

we need a Lemma to help us compute the det of this

<u>Lemma 2</u>: $\det(I + At) = 1 + \operatorname{tr}(A) \cdot t + O(t^2)$

See also Arrowsmith & Place AITDS p. 46 for a clever pf.

pf: From Atkinson AITNA p. 402

$\det(A - \lambda I) = (-1)^n \lambda^n + \operatorname{tr}(A) \lambda^{n-1} + \dots + \det(A)$

so here $\det(At + I) = t^n \det\left(A + \frac{1}{t} I\right) \qquad \lambda := -\frac{1}{t}$

$\Rightarrow (-1)^n \lambda^{-n} \det(A - \lambda I) = 1 + (-1)^n \operatorname{tr}(A) \lambda^{-1} + \hat{O}(\lambda^{-2})$

$\rightsquigarrow 1 + (-1)^{n-1} \operatorname{tr}(A) t + O(t^2)$

□

cont'd →

p. 70

We had $V(t) = \int_{D_0} \left| \det \left[d(\varphi_t)_x \right] \right| dx$

$= \int_{D_0} \left| \det \left[\underbrace{I + t\, df_x + \mathcal{O}(t^2)}_{\det = 1 + t \cdot tr(df_x) + \mathcal{O}(t^2)} \right] \right| dx$ Lemma 2.

~~det (A+B) = det A + det B~~ FALSE

$= \int_{D_0} \left| 1 + t\, tr[df_x] + \det(\mathcal{O}(t^2)) \right| dx$

Then $\frac{d}{dt} V(t) = \frac{d}{dt} \int_{D_0} |\cdots| \, dx = \int_{D_0} \left| \frac{d}{dt} (\cdots) \right| dx$ since integrand is continuous

$\Rightarrow \dot{V}(0) = \int_{D_0} \left| tr[df_x] \right| dx$ $\qquad [df_x] = \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^n}{\partial x_1} & \cdots & \frac{\partial f^n}{\partial x^n} \end{bmatrix}$

$= \int_{D_0} \left| \sum^n \frac{\partial f^i}{\partial x^i} \right| dx$

$= \int_{D_0} \left| div(f) \right| dx$ END LEMMA 1 $\square$

$\Rightarrow$ So if $div(f) = 0$, $\dot{V}(0) = 0 \Rightarrow$ Vol is const

Thm 2 $\square$

In the special case of Hamilton's Eqs

$\begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \\ \dot{p}_1 \\ \vdots \\ \dot{p}_n \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial H}{\partial p_1}(q,p) \\ \vdots \\ -\frac{\partial H}{\partial q_n} \end{bmatrix}}_{\text{"}h\text{"}}$ form " $\dot{x} = f(x)$ "

" $\vec{\nabla} \cdot \vec{f}$ " $= \nabla \cdot h = \underbrace{\sum^n \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(\frac{-\partial H}{\partial q_i}\right)}_{=0}$ since we can interchange partials

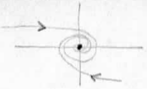
$= \sum 0$

$= 0.$

$div(h) = 0$

$\square$

$T^*M = \mathbb{R}^2$



Let $\Phi : \mathbb{R} \times T^*M \to T^*M$ be the Hamiltonian flow and a a FP.

p. 115 By def, a is asymptotically stable if Given $\varepsilon > 0$, $\exists\, \delta > 0$, $T_\varepsilon \ni$

At $t = 0$, $\forall x \in B(a, \delta)$, $\Phi^t(x) \in B(a, \varepsilon)$ $\forall\, t > T_\varepsilon$.

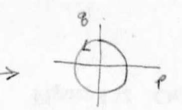
δ depends on a, not ε.

choose $\varepsilon < \delta$ and we see the flow ain't volume preserving!

[Nowhere could I find a clean def of Asymp Stab
That really says what I just said!
Maybe I'm wrong!!]

p. 187

ex 1

$$[0, 2\pi] \xrightarrow{C}$$ 

$$\begin{bmatrix} C^1(t) \\ C^2(t) \end{bmatrix} = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$$

$$\int_{C[0,2\pi]} p\,dq = \int_0^{2\pi} C^1(t)\,\dot{C}^2(t)\,dt = \int_0^{2\pi} \cos(t)\cos(t)\,dt = \pi$$

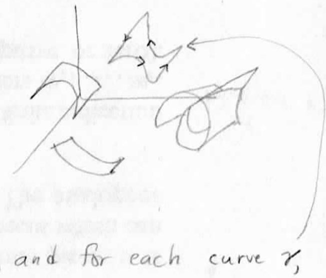
Area $= \pi r^2 = \pi 1^2$

ex 2 $\quad M$ 3-mfd in $\mathbb{R}^3$

$$\omega^1 = a_1\,dx + a_2\,dy + a_3\,dz \qquad [a_1\ a_2\ a_3]\,[\,\cdot\,]$$

$$\int_l \omega^1 = \int_0^1 a_1(t)\,\dot{l}_1(t)\,dt + a_2(t)\,\dot{l}_2(t)\,dt + a_3(t)\,\dot{l}_3(t)\,dt$$

Prob 14 $\quad \mathcal{C}$ is a 2-chain, which would have an image like:



Presumably we can apply a version of Stokes thm

$$\int_{\mathcal{C}} \sum dp^i \wedge dq^i = \int_{\partial\mathcal{C}} \sum p_i\,dq_i$$

$\partial\mathcal{C}$ is a collection of pw smooth curves, and for each curve γ, we know from above that $\int_\gamma p_i\,dq_i =$ Area of the set in $\langle p_i, q_i \rangle$-plane that is enclosed by the projection of the image of γ.

p.210

$$M \xrightarrow{\varphi} \mathbb{R}$$

we will take $\varphi = \alpha^{(i)}$ where α is a chart map.

$$\Delta : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(t,s) \longmapsto \varphi(A^t B^s x) - \varphi(B^s A^t x)$$

Really we should write $\Delta_{\varphi,x}(t,s)$ but that is too cumbersome.

Apostol MA p. 361 Taylor Thm

$$"x_1 = t" \quad "x_2 = s"$$

$$\Delta(t,s) = \underset{\substack{\parallel \\ 0}}{\Delta(0,0)} + \sum_{i=1}^{2} (D_i\Delta)_{(0,0)}\, x_i + \frac{1}{2}\sum\sum (D_{ij}\Delta)_{(0,0)}\, x_i x_j + \frac{1}{6}\sum\sum\sum (D_{ijk}\Delta)_{\xi}\, x_i x_j x_k.$$

example of calculations: $\frac{\partial\Delta}{\partial t}(t,s) = T\varphi_{A(t,B(s,x))} \cdot \frac{\partial A}{\partial t}(t, B(s,x)) - T\varphi_{B(s,A(t,x))} \cdot T(B_s)_{A(t,x)} \cdot \frac{\partial A}{\partial t}(t,x)$

For t derivs, s is fixed at 0.

Hence $\frac{\partial\Delta}{\partial t}(t,0) = T\varphi_{A(t,x)} \cdot \frac{\partial A}{\partial t}(t,x) - T\varphi_{A(t,x)} \cdot \mathrm{Id}_{A(t,x)} \cdot \frac{\partial A}{\partial t}(t,x)$

$= 0$

So we see immediately $\frac{\partial\Delta}{\partial t}(0,0) = 0$ and $\frac{\partial}{\partial t}\left(\frac{\partial\Delta}{\partial t}(t,0)\right) = 0$

so $\frac{\partial^2\Delta}{\partial t^2}(0,0) = 0$.

By symmetry, same results hold for $\frac{\partial}{\partial s}$.

Thm

$(\Rightarrow)$ For any φ, we form $\Delta_{\varphi,x}(t,s) := \varphi(A^t B^s x) - \varphi(B^s A^t x) \equiv 0$

then $\frac{\partial^2}{\partial s\partial t}\Delta(0,0) = 0 \overset{\text{Lemma 1}}{\Longrightarrow} (L_B L_A \varphi - L_A L_B \varphi)(x) = 0$

$= (L_{[A,B]}\varphi)(x) = 0$

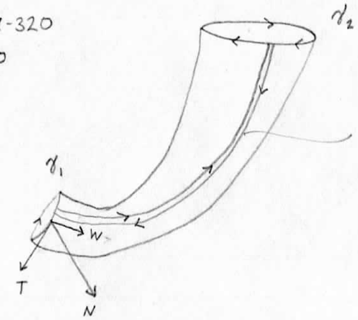
but since φ is arb, we must have $\overset{\text{Lie}}{[A,B]} = 0$.

P.234

Munkres AOM p. 319-320

p. 290

In my picture, γ_2 is oriented opposite of Arnold's version Fig 180



Imagined curve to help us orient γ_2 consistently w/ γ_1

<u>Stokes Thm</u> : $\displaystyle\int_M (\nabla\times F)\cdot N\,dA = \int_{\partial M} \vec{F}\cdot\hat{T}\,ds \overset{\text{also symbolized}}{=} \int_{\partial M} \vec{F}\cdot d\vec{s}$

"F" = V in Arnold's notation.

Here M is the 2-mfd defined as the set of pts in $\mathbb{R}^3$ that is swept out by the flowlines of $R = \mathrm{Curl}(V)$ which pass thru γ_1

So obviosly $\vec{R}(x) \in T_xM$ and $\hat{N}(x) \perp T_xM$

$$\Rightarrow\ O = \vec{R}(x)\cdot\hat{N}(x) = (\mathrm{Curl}(V))(x)\cdot N(x)$$

$$\int_M O\,dA = \int_{\gamma_1} V\cdot d\vec{s} - \int_{\gamma_2} V\cdot d\vec{s}$$

QED

<u>Prob 2</u>

I think $\omega^2 = \sum_{i=1}^{n} dp^i\wedge dq^i - \omega^1\wedge dt$ has matrix $\left[\begin{array}{c|c} \begin{array}{cc} & I \\ -I & \end{array} & 0 \\ \hline a\ b\ c \cdots z & 1 \end{array}\right]$ clearly nsing.

M (2n+1) - mfd

ω^1 1-form on M.

Since we later assume $d\omega^1$ is nsing $\}$ $\exists!$

we know from the Lemma that for every $x \in M$, $\exists!$ a vector $\Xi_x \in T_xM \ni d\omega^1_x(\Xi_x, \cdot) = O(\cdot)$

$\Rightarrow$ There is a vector field $X_{\omega^1} : M \longrightarrow TM$, $x \longmapsto \Xi_x$ $\leftarrow$ Arnold calls this the 'vortex direction'.

The streamlines of X_{ω^1} are called the 'vortex lines'.

The set of pts of M that are passed thru by streamlines of X_{ω^1} originating in γ is called the "Vortex Tube of γ".

q_0, Q_0, P_0 are all determined; we seek S_1.

CLAIM: If g is a Cannonical Trans $\triangleright$ $\det\left[D_1(g_2)_{(p_0,q_0)}\right] \neq 0$ $\Big\}$ $\Rightarrow$ In a nbhd of (Q_0, q_0) we can define a fcn S_1 $\ni$

$$\begin{cases} S_1(Q,q) = S(p,q) \quad \text{AND} \\ \dfrac{\partial S_1}{\partial q}(Q,q) = p \qquad \dfrac{\partial S_1}{\partial Q}(Q,q) = -P \end{cases}$$

$$g = (g_1, g_2)$$

$$(g_2 \times \pi_2)$$

we need to know this map is a local diffeo so that its image is an open set around (Q_0, q_0)

$$S, \quad S_1, \quad \mathbb{R}$$

pull back

$$dS_1 = \left((g_2 \times \pi_2)^{-1}\right)^* dS$$

$$\left[D(g_2\times\pi_2)_{(p_0,q_0)}\right] = \begin{bmatrix} \dfrac{\partial g_2^i}{\partial p_j} & \dfrac{\partial g_2^i}{\partial q_j} \\ 0 & I \end{bmatrix}$$

so $\det\left[D(g_2\times\pi_2)_{(p_0,q_0)}\right] = \det\left[\dfrac{\partial g_2^i}{\partial p_j}\right] = \det\left[D_1(g_2)_{(p_0,q_0)}\right] \neq 0$ (by hypoth.)

Apply Inv Fcn Thm to $g_2 \times \pi_2$ and we have the result.

NOTE relevant to p. 267

In more generality, $g = [g^1, g^2, \ldots, g^{2n}]^T$. Then instead of working with $g_2 = g^{n+1} \times \cdots \times g^{2n}$ we can work with any subset $\{i_1, \ldots, i_n\} \subset \{1, \ldots, 2n\}$ and have the map $(\underbrace{g^{i_1} \times g^{i_2} \times \cdots \times g^{i_n}}_{h_I}, \pi_2) : (p,q) \longmapsto (P_I, Q_{I_2}, q)$

claim: For at least one such index set I, $\det\left[D_p(h_I)_{(p_0,q_0)}\right] \neq 0$

pf. $Dg_{(p_0,q_0)} = \Big[\ \ \Big]_{2n \times 2n}$ $\quad D_p(h_I)$ is a $n \times n$ submatrix chosen from n rows from this half of Dg

if every such submatrix had $\det = 0$

$\Rightarrow$ $\det Dg = 0$

$\not\Leftarrow$ since g is diffeo.

$\square$

q and Q are prescribed.

$$(Q_0, q_0) \qquad S_1 \longrightarrow \mathbb{R}$$

$$q \xrightarrow{\ g\ } Q$$

THM

Given $S_1 : \mathbb{R}_Q \times \mathbb{R}_q \,(\text{nbhd of } (Q_0, q_0)) \longrightarrow \mathbb{R}$, $\det\left[\dfrac{\partial^2 S_1}{\partial Q_j \partial q_i}(Q_0, q_0)\right] \neq 0$ $\Big\}$ $\Rightarrow$ $\exists$ cannonical trans $g : \{\text{nbhd of } (p_0, q_0)\} \subset \mathbb{R}^n_p \times \mathbb{R}^n_q \longrightarrow \{\text{nbhd of } (P_0, Q_0)\} \subset \mathbb{R}^n_P \times \mathbb{R}^n_Q$

Remark: $D_q(S_1) : \mathbb{R}^n_Q \times \mathbb{R}^n_q \longrightarrow \mathcal{L}(\mathbb{R}^n \to \mathbb{R}) \cong (\mathbb{R}^n_p)^T$

$\nabla_q(S_1)$

step 1

Define $F : \mathbb{R}^n_Q \times \mathbb{R}^n_q \longrightarrow \mathbb{R}^n_p$

$$(Q, q) \longmapsto \frac{\partial S_1}{\partial q}(Q,q) = p.$$

$$\left[D_Q F_{(Q,q)}\right] = \left[\frac{\partial^2 S_1}{\partial Q_j \partial q_i}(Q,q)\right] \quad \text{so} \quad \det\left[\frac{\partial^2 S_1}{\partial Q_j \partial q_i}(Q_0, q_0)\right] \neq 0 \Rightarrow D_Q F_{(Q_0,q_0)} : \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ is an iso.}$$

Avez D.C p.28 Imp Fcn Thm: $\exists$ smooth g_1 : nbhd of $(p_0, q_0) \longrightarrow \mathbb{R}^n$

$$\boxed{g_1(p,q) = Q}$$

$$F(g_1(p,q), q) = p$$

$$(g_1, \pi_2)$$

step 2

Now we need an expression for $P = g_2(p,q)$

Define $G : \mathbb{R}^n_Q \times \mathbb{R}^n_q \longrightarrow \mathbb{R}^n_P$

$$(Q,q) \longmapsto -\frac{\partial S_1}{\partial Q}(Q,q) = P \qquad \left[\ P_i = -D_i(S_1)_{(Q,q)}\ \right]$$

but we can then define $\boxed{P := g_2(p,q)} = -D_i(S_1)(g_1(p,q), q)$

Then $g = (g_1, g_2)$ and g is a Cannonical Trans because:

$$\sum\left(p_i\,dq_i - P_i\,dQ_i\right) = \sum \frac{\partial S_1}{\partial q_i}dq_i + \frac{\partial S_1}{\partial Q_i}dQ_i = d(S_1)_{(Q,q)} \quad g_1(p,q)$$

$$= dS_{(p,q)}$$

See p.258 $\square$

$$(g_1, \pi_2)(\cdot,\cdot) \qquad S_1 \searrow \mathbb{R} \swarrow S$$

d commutes w/ pull back

$$dS = (g_1, \pi_2)^* dS_1$$

p. 267

$$\theta_1 - g^*\theta_2 = dS$$

For some fcn S depending only on $\langle p,q\rangle$

KEY IDEA: we must have $\sum p_j\,dq_j - P_j\,dQ_j = dS_{(p,q)}$ ⊛

in order for $\langle p,q\rangle \xrightarrow{g} \langle P,Q\rangle$ to be Cannonical Trans.

▷ If we are given $S = S_1(q,Q)$ then we are directly able

to use the dq, dQ form of $\theta_1 - g^*\theta_2$:

$$p\,dq - P\,dQ$$

$$d(S_1)_{(q,Q)} = \frac{\partial S_1}{\partial q} \qquad \frac{\partial S_1}{\partial Q}$$

▷ However, if we are given $S_2(q,P)$ then

$$d(S_2)_{(q,P)} = \frac{\partial S_2}{\partial q}dq + \frac{\partial S_2}{\partial P}dP$$

So we must convert ⊛ to a form that explicitly features dq and dP [however it still must be a fcn of p,q: $P(p,q)$, $Q(p,q)$].

By adding $d(PQ)$ to both sides in ⊛ we get

$$⊛ \iff p\,dq + Q\,dP = d(\underbrace{PQ + S}_{S_2(q,P)})$$

[This could be glossing over an important detail

see Moser's paper 1976 Birkhoff my p. 489–490 work thru]

The map $\langle p,q\rangle \xrightarrow{g} \langle P,Q\rangle$ is a CT because:

$$d(S_2)_{(q,P)} = \frac{\partial S_2}{\partial q}dq + \frac{\partial S_2}{\partial P}dP = p\,dq + Q\,dP$$

then

$$dS_2 - d(PQ) = p\,dq + Q\,dP - d(PQ) \qquad \sum P_jQ_j$$

$$d(\underbrace{S_2 - PQ}_{S}) = p\,dq - P\,dQ$$

Since we have determined $P = g_1(p,q)$, $Q = g_2(p,q)$, S depends only on (p,q)

and therefore $dS_{(p,q)} = p\,dq - P\,dQ$ The LHS is the total differential of a fcn depending only on $\langle p,q\rangle$

$\Rightarrow g$ is CT. □

---

we want a generating fcn for $I: \langle p,q\rangle \longmapsto \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} P \\ Q \end{bmatrix}$.

we could try ① $S(q,Q)$, ② $S(p,Q)$, ③ $S(q,P)$, ④ $S(p,P)$

[or even a mixed combo of the type on p. 268, but I won't consider that here].

(1) Following p. 258 BOPP, we must have $\langle p,q\rangle \xrightarrow{\pi_2 \times g_2} \langle q,Q\rangle$ a diffeo in a nbhd of $\langle q_0,Q_0\rangle$.

Here $D(\pi_2 \times g_2) = \left[\begin{array}{c|c} \frac{\partial q}{\partial p} & \frac{\partial q}{\partial q} \\ \hline \frac{\partial Q}{\partial p} & \frac{\partial Q}{\partial q} \end{array}\right] = \left[\begin{array}{c|c} 0 & I \\ \hline 0 & I \end{array}\right]$ not invertible.

(2) $\langle p,q\rangle \longmapsto \begin{bmatrix} p \\ Q \end{bmatrix} = q$ $\quad \left[\begin{array}{c|c} \frac{\partial p}{\partial p} & \frac{\partial p}{\partial q} \\ \hline \frac{\partial Q}{\partial p} & \frac{\partial Q}{\partial q} \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]$ This works

(3) $\langle p,q\rangle \longmapsto \begin{bmatrix} q \\ P \end{bmatrix} = p$ $\quad \left[\begin{array}{c|c} & \\ \hline & \end{array}\right] = \left[\begin{array}{c|c} & I \\ \hline I & \end{array}\right]$ This works

(4) $\langle p,q\rangle \longmapsto \begin{bmatrix} p \\ P \end{bmatrix}$ $\quad \left[\begin{array}{c|c} I & \\ \hline I & \end{array}\right]$ fails.

So we may use $S(p,Q)$ or $S(q,P)$.

For the first, a viable fcn is $S(p,Q) = \sum p_jQ_j$ THEN

$$q_j := \frac{\partial S}{\partial p_j} = Q_j \qquad P_j = \frac{\partial S}{\partial Q_j} = p_j \qquad \text{Id}$$

For the 2nd type, take $S(q,P) = \sum q_jP_j$

$$\Rightarrow p_j := \frac{\partial S}{\partial q_j} = P_j \qquad Q_j := \frac{\partial S}{\partial P_j} = q_j \qquad \text{Id}$$

□

P. 272

Another variant of (2) is:

If $\mathcal{M}_f$ is Conn and each X_{dF_i} is <u>Complete</u> vf

$$\Rightarrow \mathcal{M}_f \text{ diffeo to } T^k \times \mathbb{R}^{n-k}$$

$\boxed{\text{see A\&M p. 113}}$

Notes on the pf of Lemma 1:

vfs X, Y are said to <u>commute</u> if Lie bracket vanishes: $[X,Y]^L = 0$ $\boxed{\text{P. 211}}$

$$[X,Y]^L = 0 = -[Y,X]^L.$$

Here we know that $[X_{dF_i}, X_{dF_j}]^L = 0$ because

$\boxed{\text{L\&S P. 519–520}}$ $[X_{dF}, X_{dG}]^L = X_{d\{F,G\}^P} = X_{d0} = 0$ since we have an isomorphism from space of 1-forms to vfs.

we know that $\mathcal{M}_f$ is invariant under the flows inf generated by the vfs X_{dF_i} because:

$$\mathcal{M}_f = \tilde{\sigma}^{-1}(f), \quad \ker(d\tilde{\sigma}_x) = T_x(\mathcal{M}_f)$$ $\boxed{\text{Guillemin \& Pollack Diff TOP P. 24}}$

$\boxed{\text{Thirring P. 101}}$ $dF_i(X_{dF_j}) = \overset{\text{Lie deriv}}{L_{X_{dF_j}}}(F_i) = \{F_i, F_j\}^P = 0$ since in Involution, this holds $\forall\, i,j$

$$\Rightarrow X_{dF_i}(x) \in \ker(d\tilde{\sigma}_x) = T_x(\mathcal{M}_f)$$

So the flowlines are restricted to $\mathcal{M}_f$

□

p. 278

$\Gamma_{x_0} = \left\{ \begin{matrix} \text{all} \\ t \in \mathbb{R}^n \end{matrix} \middle| \; G_{x_0}(t) = x_0 \right\}$ additive subgroup of $\mathbb{R}^n$
discrete topology.

This is a group of covering transforms of $\mathbb{R}^n$ when it acts on $\mathbb{R}^n$ by translations and it preserves the lattice pts.

$\boxed{\text{Boothby AITDMARG P. 102 Thm 9.3}}$

$$\theta : \Gamma_{x_0} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$(t, x) \longmapsto x + t$$

It also acts transitively on $G_{x_0}^{-1}(p)$ for any $p \in M$.

$$\begin{tikzcd} & & \mathbb{R}^n \arrow[dl, "\rho"'] \arrow[d, "G_{x_0}"] \\ T^k \times \mathbb{R}^{n-k} \arrow[r, "\beta", "\text{diffeo}"'] & \mathbb{R}^n/\Gamma_{x_0} \arrow[r, "\pi,\ \text{diffeo}"] & M \end{tikzcd}$$

Note that $\mathbb{R}^n / \mathbb{Z}^k \times \{0\}$ diffeo to $T^k \times \mathbb{R}^{n-k}$

$n-k$ must be zero since M is cpt.

Prob 11 Since $T^n = S^1 \times \dots \times S^1$ we have a cannonical inverse chart, given an inverse chart for S^1 $\varphi_1^{-1} : (-\varepsilon, \varepsilon) \longrightarrow \overset{S^1}{U}$.

Then $\varphi : \overset{T^n}{U_1 \times \dots \times U_n} \longrightarrow \mathbb{R}^n$
$p \longmapsto \langle \varphi^1(p), \dots, \varphi^n(p) \rangle$ and $\left\{ \frac{\partial}{\partial \varphi^i} \right\}$ is a basis for $T_p(T^n)$.

we have the hamiltonian vf X_{dH} on $\mathcal{M}_f$ and the diffeo $\beta : T^n \longrightarrow \mathcal{M}_f$

$\beta^* X_{dH}$, X_{dH}

Pull it back to T^n and express it wrt basis $\left\{ \frac{\partial}{\partial \varphi^i} \right\}$:

$$\beta^* X_{dH} = \omega_1 \frac{\partial}{\partial \varphi^1} + \dots + \omega_n \frac{\partial}{\partial \varphi^n}$$ [ω's are not related to our 2-form ω^2]

Then we get the cond periodic motion defined by
$$\dot{\varphi}_i = \omega_i \quad i = 1 \dots n.$$

$\square$