Summary of Main Ideas here:

$$D_t : \underset{\substack{\text{moving frame}\\ \text{co-ord sys}}}{\mathbb{R}^3_K} \longrightarrow \underset{\text{fixed sys}}{\mathbb{R}^3_k}$$

$$Q(t) \longmapsto B_t Q(t) + r(t) = q(t)$$

where B_t is O.N. linear map (rotation)

$$q(t) = B_t Q(t) + r(t)$$

$$\Rightarrow \dot{q} = \dot{B}Q + B\dot{Q} + \dot{r}$$

▷ If there is no rotation (B is indep of t)

$$\Rightarrow \dot{q} = B\dot{Q} + \dot{r} \quad \text{which Arnold writes as}$$

$$\text{“} v = v' + v_0 \text{”}$$

▷ Another special case: particle at rest in moving sys

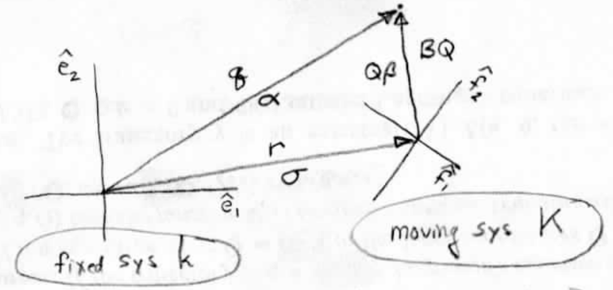
$\dot{Q} = 0$ and just take $r = 0$ [origins coincide]

$$\Rightarrow \dot{q} = \dot{B}Q$$

called "transferred rotation" by Arnold

p.126 **Thm** $\dot{q} = \dot{B}Q \Longrightarrow \exists\, \vec{\omega}_t \in \mathbb{R}^3_k \ni \dot{q} = \omega \times q$

pf on next sheet



My notation is the curve α is the trajectory of the particle in fixed sys $\quad Q_t$ = ON rotation matrix

$$\alpha = Q_t \beta + \sigma$$

where β is traj wrt moving co-ord sys

σ is traj of origin of moving sys wrt fixed sys.

<u>My argument 1999:</u>

$$\vec{f}_i = (\vec{f}_i)_e$$

$$Q = [\hat{f}_1\ \hat{f}_2\ \hat{f}_3]$$

$$\beta = Q^T \alpha$$

$$Q : \mathbb{R}^3_f \longrightarrow \mathbb{R}^3_c \quad \left[\text{Arnold } \mathbb{R}^3_K \to \mathbb{R}^3_k\right]$$

$$\beta(t) = \underbrace{Q_t^T(\alpha(t))}_{G(t,\alpha(t))} \longleftarrow \text{linear in } 2^{\text{nd}} \text{ arg.}$$

$$\dot{\beta} = D_1 G_{t,\alpha}(1) + D_2 G_{t,\alpha}(\dot{\alpha})$$

$$= \dot{Q}_t^T \alpha + Q_t^T \dot{\alpha}$$

Lemma

$\dot{Q}^T = -WQ^T$

for skew symm matrix $W : W^T = -W$

$\dot{Q} = QW$

$Q^T \dot{Q} = W$

$$= -W\underbrace{Q^T\alpha}_{\beta} + Q^T\dot{\alpha}$$

$$\dot{\beta} = -W\beta + Q^T\dot{\alpha}$$

$$Q^T\dot{\alpha} = \dot{\beta} + W\beta$$

$$\dot{\alpha} = Q(\dot{\beta} + W\beta) = Q(\dot{\beta} + \Omega \times \beta)$$

since $W(\cdot) = \vec{\Omega} \times (\cdot)$

Arnold would write

$$\dot\alpha = Q(\dot\beta + \vec\Omega \times \vec\beta)$$

we must recognize this as ω_f (it is the representation in the rotating sys).

$$\dot\alpha = Q_t(\dot\beta + \omega \times \beta) + \dot\sigma$$

"Transferred rotation" $\dot\beta = 0$ $\quad \dot\sigma = 0$

$$\dot\alpha = Q(\omega_f \times \beta)$$

$$= Q\omega_f \times Q\beta$$

by the formula $Q(a \times b) = \frac{1}{\det(Q^{-1})^T}\left[(Q^{-1})^T a \times (Q^{-1})^T b\right]$

$Q^T = Q^{-1}$ so $(Q^{-1})^T = Q$

$\det Q = +1$ (rotation)

$$\Rightarrow Q(a \times b) = \frac{1}{1}\left[Qa \times Qb\right]$$

$$\boxed{\dot{\vec\alpha} = \vec\omega_e \times \vec\alpha}$$

$$\underset{\text{"}\omega_f\text{"}}{Q\omega} = \underset{\text{"}\omega_e\text{"}}{\Omega}$$

Arnol'd has $\Omega = B^{-1}\omega$

Arnold's pf

Thm $\dot q = \dot B Q \Rightarrow \dot q = \omega \times q$

$\hat\omega$ is instantaneous ang velocity

$$\dot q = \dot B Q \qquad Q = B^{-1} q$$

$$= \underbrace{\dot B B^{-1}}_{A} q \quad \text{where } A \cdot k \longrightarrow k$$

Lemma $A^T = -A$ skew symm

Lemma A skew symm $\Rightarrow \exists$ vector $\vec a \ni A(\cdot) = \vec a \times (\cdot)$

Then for our $A = \dot B B^{-1}$ we call the vector $\vec\omega$

$$\Rightarrow \dot q = \omega \times q$$

$\square$

NOTE I worked with $\dot Q = QW$ but the analog for Arnold would be $\dot B = AB$

back / front

p.130

Given Newtonian eq for motion of a particle
in $\mathbb{R}^3_k$: $m\ddot{q} = f(q,\dot{q})$ [inertial sys]
How does this transform into the moving sys $\mathbb{R}^3_K$?
Answer: $m\ddot{Q} = F(Q,\dot{Q}) - m\dot{\Omega}\times Q - 2m\Omega\times\dot{Q} - m\Omega\times(\Omega\times Q)$

Arnold has $m\ddot{q} = f(q,\dot{q})$

$$q = BQ$$
$$\dot{q} = \dot{B}Q + B\dot{Q}$$
$$= B\left[\Omega\times Q + \dot{Q}\right]$$
$$\ddot{q} = B\left[\ddot{Q} + \dot{\Omega}\times Q + 2\Omega\times\dot{Q} + \Omega\times(\Omega\times Q)\right]$$

mult both sides by m

Then
$$B^T(m\ddot{q}) = m\ddot{Q} + m\dot{\Omega}\times Q + 2m\Omega\times\dot{Q} + m\Omega\times(\Omega\times Q)$$
$$\parallel$$
$$f(q,\dot{q})$$
$$\parallel \quad q = BQ$$
$$f\left(BQ, \tfrac{d}{dt}(BQ)\right)$$

$$\underbrace{B^T\cdot f\left(BQ, \tfrac{d}{dt}(BQ)\right)}_{F(Q,\dot{Q})} - m\dot{\Omega}\times Q - 2m\Omega\times\dot{Q} - m\Omega\times(\Omega\times Q) = m\ddot{Q}$$

So if $f(q,\dot{q}) \equiv 0$ (no force)
then we see $F(Q,\dot{Q}) = 0$ but all these terms survive

So we can see how the rotating sys is not an inertial frame.

P. 136-137 single particle

we know ang velocity $\dot{q} = \omega \times q$

ang momentum $\vec{L} = \vec{r} \times m\vec{v}$

Arnold: $\vec{m} = q \times m\dot{q}$
$= q \times m(\omega \times q)$

apply B_t to both sides: $B\vec{m} = B(q \times m(\omega \times q))$

B O.N. rotation
det B = +1

$$= Bq \times mB(\omega \times q)$$
$$= Q \times m(B\omega \times Bq)$$
$$\vec{M} = Q \times m(\Omega \times Q)$$

Now Arnold defines single pt inertia operator:

In rotating system

$$A_{m,Q} : \mathbb{R}^3_K \longrightarrow \mathbb{R}^3_K$$
$$\Omega \longmapsto m(Q \times (\Omega \times Q))$$

Thus $A(\Omega) = M$
$[\mathbb{I}\Omega = \vec{L}]$

$\parallel$ vector relation

$a \times (b \times c) = \vec{b}(a \cdot c) - \vec{c}(a \cdot b)$

$$= m\vec{\Omega}(Q^TQ - \vec{Q}(Q^T\Omega))$$
$$= \underbrace{m[Q^TQI - QQ^T]\vec{\Omega}}$$

This is the physics book form of inertia tensor $\mathbb{I}$

Thm $A_{m,Q}$ is linear and symmetric ($A^T = A$).

show Symm Arnold's way: $\langle A\Omega, \Psi \rangle \stackrel{?}{=} \langle \Omega, A\Psi \rangle$

$$\langle Q \times (\Omega \times Q), \Psi \rangle \stackrel{?}{=} \langle \Omega, Q \times (\Psi \times Q) \rangle$$

def of triple prod

$$= \det \begin{vmatrix} - \Psi - \\ - Q - \\ - \Omega \times Q - \end{vmatrix}$$

2 row exchanges: $(-1)^2 = 1$

$$= \begin{vmatrix} \Omega \times Q \\ \Psi \\ Q \end{vmatrix}$$

$$= \langle \Omega \times Q, \Psi \times Q \rangle \stackrel{\text{symm of trip}}{=} \langle \Psi \times Q, \Omega \times Q \rangle \stackrel{\text{def of trip P}}{=} \begin{vmatrix} \Psi \times Q \\ \Omega \\ Q \end{vmatrix}$$

2 row exch

$$= \begin{vmatrix} \Omega \\ Q \\ \Psi \times Q \end{vmatrix} = \langle \Omega, Q \times (\Psi \times Q) \rangle$$

□

COR Quadratic form $\frac{1}{2}\langle A_{m,Q}\Omega, \Omega \rangle = \frac{1}{2} m \|v\|^2$ KE of single revolving particle

pf. $\frac{1}{2}\langle A_{m,Q}\Omega, \Omega \rangle = \frac{m}{2}\langle \Omega \times Q, \Omega \times Q \rangle$

$$= \frac{m}{2}\|V\|^2 \qquad V = \Omega \times Q$$
$$= \frac{m}{2}\|B^{-1}v\|^2$$
$$= \frac{m}{2}\|v^2\| \text{ since B is O.N. matrix and preserves lengths}$$ □

Thm Angular Momentum $\vec{M}_0$ of RB B wrt fixed pt O depends linearly on ang vel $\vec{\Omega}$.

i.e. $\exists$ symm linear map $A^B : \mathbb{R}^3_K \longrightarrow \mathbb{R}^3_K$
$\vec{\Omega} \longmapsto \vec{M}_0$

& KE $T_B = \frac{1}{2}\langle A\Omega, \Omega \rangle$

pf Let $B = \bigcup^N Q_i$ finite union of mass pts

① Then $\vec{M} = \sum^N M_{Q_i} = \sum^N A_{m_i,Q_i}(\Omega) = (\sum A_i)(\Omega) = A(\Omega)$

② A_B is symm because each A_i is:

$$\langle A_B x, y \rangle = \langle \sum^N A_i x, y \rangle = \sum \langle A_i x, y \rangle = \sum \langle x, A_i y \rangle$$
$$= \langle x, Ay \rangle.$$

③ KE $T_B = \sum T_i = \frac{1}{2}\sum^N \langle A_{m,Q_i}\Omega, \Omega \rangle$

$$= \frac{1}{2}\langle \sum A_i \Omega, \Omega \rangle$$
$$= \frac{1}{2}\langle A_B \Omega, \Omega \rangle$$

□

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Derive Euler's Eqs

First transform angular momentum into the rotating sys: $\sigma = 0$ Both systems have same origin

In the fixed sys (well, any inertial frame)

$$\vec{L}_0 = \sum r_i \times m_i \dot{r}_i = \sum \alpha_i \times m_i \dot{\alpha}_i$$

physics notation

$$= \sum m_i [Q\beta_i] \times [Q(\underbrace{\dot{\beta}_i}_{=0} + \Omega \times \beta_i)]$$

0 in frame rotating with RB

$$= \sum m_i (Q\beta_i) \times (Q\,\Omega \times \beta_i)$$

$Q(a \times b) = Qa \times Qb$

Q O.N; $\det Q = +1$

$$= \sum m_i Q(\beta_i \times (\Omega \times \beta_i))$$

$$= Q \sum m_i (\beta_i \times (\Omega \times \beta_i))$$

$$\Rightarrow \underset{\substack{\| \\ L_0^f}}{Q^T L_0} = \underbrace{\sum m_i (\beta_i \times (\Omega \times \beta_i))}$$

This is Arnold's A operator

It is the inertia tensor in rotating sys

I_0 has same form in fixed and rot sys!

ie $L_0^e = Q L_0^f$

or in Arnold's notation: $\vec{m} = B\vec{M}$

Balance of Ang Momentum thm: $\frac{d}{dt}\vec{L}_0 = \vec{T}$

Arnold: $\frac{d}{dt}\vec{m} = \vec{n}$

$$\vec{n} = \frac{d}{dt}(\vec{m}) = \frac{d}{dt}(BM) = \dot{B}M + B\dot{M}$$

$$= B(\Omega \times M + \dot{M})$$

$$\Rightarrow B^T \vec{n} = \dot{M} + \Omega \times M$$

$$\vec{N} = \dot{M} + \Omega \times M \quad \text{Euler's Eq}$$

If no external torques: $\vec{n} = 0 \Rightarrow \vec{N} = 0$

$$\Rightarrow \dot{M} = M \times \Omega$$

In Co-ords, in Princ Axes sys: $A = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$

$$\vec{M} = A\Omega = \begin{bmatrix} \lambda_1 \Omega_1 \\ \lambda_2 \Omega_2 \\ \lambda_3 \Omega_3 \end{bmatrix}$$

$$\dot{M} = M \times \Omega$$

$$\begin{bmatrix} \dot{M}_1 \\ \dot{M}_2 \\ \dot{M}_3 \end{bmatrix} = \begin{vmatrix} i & j & k \\ M_1 & M_2 & M_3 \\ \Omega_1 & \Omega_2 & \Omega_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ \lambda_1\Omega_1 & \lambda_2\Omega_2 & \lambda_3\Omega_3 \\ \Omega_1 & \Omega_2 & \Omega_3 \end{vmatrix} = \begin{bmatrix} \lambda_2\Omega_2\Omega_3 - \lambda_3\Omega_3\Omega_2 \\ -(\lambda_1\Omega_1\Omega_3 - \lambda_3\Omega_3\Omega_1) \\ \lambda_1\Omega_1\Omega_2 - \lambda_2\Omega_2\Omega_1 \end{bmatrix}$$

$\square$